

0. Warm-up

Recall the theta function with characteristics $\alpha, \beta \in \mathbb{R}$, namely

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau, z) = \sum_{m \in \mathbb{Z}} e^{\pi i (m + \alpha)^2 \tau + 2\pi i (m + \alpha)(z + \beta)}$$

with $\tau \in \mathbb{H}$, $z \in \mathbb{C}$. We have the following shorthand notation

$$(\alpha, \beta) = \begin{cases} (0, 0) & \rightsquigarrow \vartheta_{00}(\tau, z), \\ (0, 1) & \rightsquigarrow \vartheta_{01}(\tau, z), \\ (1, 0) & \rightsquigarrow \vartheta_{10}(\tau, z), \\ (1, 1) & \rightsquigarrow \vartheta_{11}(\tau, z). \end{cases}$$

We then have for the elliptic curve $E_\tau = \mathbb{C} / (\mathbb{Z} \oplus \mathbb{Z}\tau)$ the relation

$$\int_{E_\tau} \log |\vartheta_{11}(\tau, z)|^2 \frac{dz \wedge d\bar{z}}{Im \tau} = \log \|\eta(\tau)\|^2.$$

The proof simply uses the relation

$$\int_{E_\tau} \log |\vartheta_{11}(\tau, z)|^2 \frac{dz \wedge d\bar{z}}{Im \tau} = \int_0^1 \int_0^1 \log \left| \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau, 0) \right|^2 d\alpha d\beta.$$

Since the latter is harmonic and transforms like a modular form it equals $\log |F(\tau)|$ with a modular form F .

1. Digression on currents.

Let $X/\mathbb{C}$ be an n -dim. projective complex manifold. A current of type (p,p) on X is a linear functional on the space of smooth $(n-p, n-p)$ -forms $A^{n-p, n-p}(X)$ of X .

Examples.

i) Smooth currents: letting $\omega \in A^{p,p}(X)$, its associated smooth current $[\omega]$ is defined by

$$[\omega](\eta) := \int_X \omega \wedge \eta.$$

ii) Dirac currents: let $Z \subset X$ be a p -codim. analytic subvariety (cycle), then its associated Dirac current δ_Z is defined by

$$\delta_Z(\eta) = \int_Z \eta.$$

iii) Green's currents: let $Z \subset X$ be as before; then, the Green's currents g_Z associated to Z is characterized by the differential equation

$$dd^c g_Z + \delta_Z = [\omega_Z]$$

for suitable differential form ω_Z ; it has type $(p-1, p-1)$. Green's currents exist by the $\partial\bar{\partial}$ -lemma.

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An example of a Green's current (or, rather, of a Green's function):

Let $\bar{L} = (L, \|\cdot\|)$ be a holomorphic line bundle on X equipped with a smooth hermitian metric $\|\cdot\|$, in short, a hermitian line bundle. Let $0 \neq s$ be a rational section of L and $D = \text{div}(s)$. The Green's function g_D for D is now given as $g_D = -\log \|s\|^2$, in fact, we have

$$dd^c(-\log \|s\|^2) + \delta_D = c_1(\bar{L}) = \text{1st Chern form.}$$

Ster-product. Let Z_1, Z_2 be two p_1, p_2 -codim. cycles on X and g_{Z_1}, g_{Z_2} its associated Green's function. We assume that Z_1, Z_2 intersect properly. Then, the ster-product $g_{Z_1} * g_{Z_2}$ of g_{Z_1} and g_{Z_2} is defined as

$$g_{Z_1} * g_{Z_2} = g_{Z_1} \wedge \delta_{Z_2} + g_{Z_2} \wedge [\omega_{Z_1}];$$

it is the Green's current for the subvariety $Z_1 \cdot Z_2$ of Z_1, Z_2 .

Functoriality. Let X, Y be projective complex varieties and $f: X \rightarrow Y$ a smooth morphism, of relative dimension d .

If $Z \subset X$ is a p -codim. cycle such that $\dim f(Z) = \dim Y$, then

$$f_* g_Z = \int_{X/Y} g_Z \quad (\text{integration along the fibers of } f)$$

constitutes a Green's current for $f_* Z = f(Z)$; if $\text{codim}(Z) = p$, i.e., g_Z is of type $(p-1, p-1)$, then $f_* g_Z$ is of type $(p-d-1, p-d-1)$.

2. Another interpretation of $\log \|\Delta\|$ and generalizations.

Let $A_{n,\theta}$ denote the moduli space of n -dim. principally polarized abelian varieties with a fixed θ divisor, and let $\pi_n: B_{n,\theta} \rightarrow A_{n,\theta}$ be the corresponding universal abelian variety. Let $\bar{L} = (L, \|\cdot\|)$ with $L = \mathcal{O}_{B_{n,\theta}}(\theta)$ be the line bundle associated to the (fixed) theta divisor on $B_{n,\theta}$ equipped with the natural translation invariant metric; in fact, we will work with $L^{\otimes 4}$, but we'll pretend to work with L .

For $n=1$, we have

$$\pi_1: B_{1,\theta} = \Gamma(2) \times \mathbb{Z}^2 \setminus \mathbb{H} \times \mathbb{C} \rightarrow \Gamma(2) \setminus \mathbb{H} = A_{1,\theta}.$$

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A basis for $H^0(B_{1,0}, L)$ (or, rather, $L^{\otimes 4}$) is given by $\vartheta_{00}, \vartheta_{01}, \vartheta_{10}, \vartheta_{11}$. On the universal elliptic curve $B_{1,0}$, we now have the Green's functions

$$g_{00} := -\log \|\vartheta_{00}(\tau, z)\|^2,$$

$$g_{01} := -\log \|\vartheta_{01}(\tau, z)\|^2,$$

$$g_{10} := -\log \|\vartheta_{10}(\tau, z)\|^2,$$

$$g_{11} := -\log \|\vartheta_{11}(\tau, z)\|^2.$$

We are interested in calculating the $*$ -products

$$\pi_{1*}(g_{11} * g_{00}), \pi_{1*}(g_{11} * g_{01}), \pi_{1*}(g_{11} * g_{10});$$

we will calculate $\pi_{1*}(g_{11} * g_{00})$ to illustrate the procedure; the other two quantities are computed similarly. By the very definition, we have

$$\begin{aligned} \pi_{1*}(g_{11} * g_{00}) &= \pi_{1*}(g_{11} \wedge \omega_{00} + g_{00} \wedge \delta_{11}) = \\ &= \int_{E_\tau} \log \|\vartheta_{11}(\tau, z)\|^2 \wedge c_1(L) - \log \|\vartheta_{00}(\tau, 0)\|^2 = \\ &= -\log \|\eta(\tau)\|^2 - \log \|\vartheta_{00}(\tau, 0)\|^2. \end{aligned}$$

Using Jacobi's triple identity, this yields

$$\begin{aligned} \pi_{1*}(g_{11} * g_{00} + g_{11} * g_{01} + g_{11} * g_{10}) &= \\ &= -6 \log \|\eta(\tau)\|^2 = -\frac{1}{4} \log \|\Delta(\tau)\|^2. \end{aligned}$$

In general, we have the following result.

Theorem. Consider the universal abelian variety $\pi_n: B_{n,g} \rightarrow A_{n,g}$ together with the line bundle $L = \mathcal{O}_{B_{n,g}}(\Theta)$ (in fact, again we need to work with a sufficiently high power of L). Then, there exist $(n+1)$ sections $s_1, \dots, s_{n+1} \in \Gamma(B_{n,g}, L)$ (whose divisors intersect pairwise properly) such that

$$\int_{A_\tau} g_1 * \dots * g_{n+1} = -\log \|F_n(\tau)\|^2,$$

where $g_j = -\log \|s_j\|^2$ ($j=1, \dots, n+1$) and F_n is a Siegel modular form for $T_{1,2}$ of weight $(n+1)!/2$.

Idea of proof. We have

$$dd^c(g_1 * \dots * g_{n+1}) + \sum_{\text{div}(s_1) \dots \text{div}(s_{n+1})} = c_1(\bar{L})^{n+1}.$$

After applying π_* and interchanging π_* with dd^c , we get

$$dd^c(\pi_*(g_1 * \dots * g_{n+1})) + \sum_{\pi_*(\text{div}(s_1) \dots \text{div}(s_{n+1}))} = \pi_*(c_1(\bar{L})^{n+1}).$$

Since $\pi_* L$ is trivial, the Riemann-Roch theorem (at the level of cohomology classes) gives

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$$\mathrm{ch}(\pi_* L) = \pi_* (\mathrm{ch}(L) \cdot \mathrm{Td}(\pi)) ,$$

hence

$$0 = c_1(\pi_* L) = \pi_* \left(\frac{c_1(L)^{n+1}}{(n+1)!} - \frac{c_1(L)^n}{n!} \cdot \frac{c_1(\Omega_{B_2, \theta}^1/A_2, \theta)}{2} \right)$$

which gives

$$\pi_* (c_1(L)^{n+1}) = \frac{(n+1)!}{2} c_1(\omega_n)$$

with the Hodge bundle $\omega_n = \pi_* (\det(\Omega_{B_2, \theta}^1/A_2, \theta))$.

Example for $n=2$. Consider $\pi_2 : B_{2, \theta} \rightarrow A_{2, \theta}$ with L as above. For a generic $\tau \in A_{2, \theta}$ (i.e., $\tau \notin \mathcal{H}$ = Humbert surface $\subset A_{2, \theta}$) and $P \in A_\tau$, put $s_P(\tau, z) := \mathcal{O}(\tau, z - P)$.

It can then be shown that for suitable choices of 2-torsion points $P, Q, R \in A_\tau$ (see below), one obtains pairwise properly intersecting sections s_P, s_Q, s_R of L such that the corresponding Green's functions g_P, g_Q, g_R satisfy

$$\sum_{P, Q, R} \int_{A_\tau} g_P * g_Q * g_R = -576 \log \|X_{10}(\tau)\|^2;$$

here P runs through the six odd 2-torsion points of A_τ , Q runs through the ten 2-torsion points not on $\mathrm{div}(s_P)$, while R runs finally through the remaining ~~four~~ 2-torsion points not on $\mathrm{div}(s_P) \cap \mathrm{div}(s_Q)$.

3. Explanation of the two different appearances of $\log \|\Delta\|$.

We return to the notations of section 1. The p -th arithmetic Chow group is then defined as

$$\hat{CH}^p(X) := \{ (Z, g_Z) \mid \begin{array}{l} Z = p\text{-codim. cycle} \\ g_Z = \text{Green's current for } Z \end{array} \}$$

The direct sum $\bigoplus_{p \geq 0} \hat{CH}^p(X)$ has a product structure given by

$$\hat{CH}^p(X) \times \hat{CH}^q(X) \rightarrow \hat{CH}^{p+q}(X)$$

$$((Z_1, g_{Z_1}), (Z_2, g_{Z_2})) \mapsto (Z_1 \cdot Z_2, g_{Z_1} * g_{Z_2}).$$

Furthermore, there is a push-forward for $f: X \rightarrow Y$, namely

$$f_*: \hat{CH}^p(X) \rightarrow \hat{CH}^{p-d}(Y)$$

$$(Z, g_Z) \mapsto (f_*(Z), f_* g_Z).$$

Finally, for hermitian line bundles $\bar{L} = (L, \|\cdot\|)$, we have arithmetic Chern classes

$$\hat{ch}(\bar{L}) = \sum_{k=0}^{\infty} \frac{\hat{c}_k(\bar{L})^k}{k!},$$

where $\hat{c}_1(\bar{L}) = (\text{div}(s), -\log \|s\|^2) \in \hat{CH}^1(X)$.

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Arithmetische Riemann-Roch Theorem.

With the above notations, we have

$$\hat{ch}(f_* L, \|\cdot\|_Q) = f_* (\hat{ch}(\bar{L}) \cdot \hat{Td}(\bar{T}_f)) \\ (+ \text{top. corr. term}).$$

Application to $\pi: B_{n,\Theta} \rightarrow A_{n,\Theta}$ in two situations

1.) $L = \mathcal{O}_{B_{n,\Theta}}$ $\|\cdot\| = \text{trivial}$:

$$\hat{ch}(\pi_* \mathcal{O}_{B_{n,\Theta}}, \|\cdot\|_Q) = \pi_* (\hat{ch}(\bar{\mathcal{O}}_{B_{n,\Theta}}) \hat{Td}(\bar{T}_\pi)), \\ \text{i.e., in degree 1}$$

$$\hat{c}_1(\det \pi_* \mathcal{O}_{B_{n,\Theta}}, \|\cdot\|_Q) = -\frac{\hat{c}_1(\bar{\omega}_n)}{2}.$$

2.) $L = \mathcal{O}_{B_{n,\Theta}}(\Theta)$, $\|\cdot\| = \text{translation invariant}$

$$\hat{ch}(\pi_* L, \|\cdot\|_Q) = \pi_* (\hat{ch}(L) \hat{Td}(\bar{T}_\pi)),$$

i.e., in degree 1 (note $\pi_* L = \mathcal{O}_{A_{n,\Theta}}$)

$$\hat{c}_1(\mathcal{O}_{A_{n,\Theta}}, \|\cdot\|_Q) = 0 = \pi_* \left(\frac{\hat{c}_1(\bar{L})^{n+1}}{(n+1)!} \right) - \frac{\hat{c}_1(\bar{\omega}_n)}{2}$$

$$\Rightarrow \hat{c}_1(\det \pi_* \mathcal{O}_{B_{n,\Theta}}, \|\cdot\|_Q) = -\pi_* \left(\frac{\hat{c}_1(\bar{L})^{n+1}}{(n+1)!} \right)$$

$\Rightarrow \|\cdot\|_Q$ als $*$ -Product of Green's functions.