# Cyclic Orbifolds of the Leech Lattice Vertex Operator Algebra

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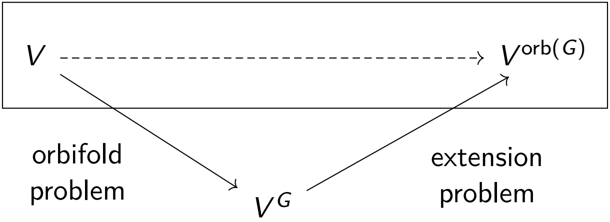
# Motivation

Three important problems in the structure theory of vertex operator algebras:

- *Orbifold problem*: Properties of $V^G$ for a vertex operator algebra $V$ and a group $G$ of automorphisms of $V$.
- *Extension problem*: Build vertex operator algebra $V$ from the modules of a smaller vertex operator algebra.
- *Classification problem*: Classify all vertex operator algebras $V$ with given properties.

# Motivation

classification problem



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Section 1

# Niemeier Lattices

# Lattices

- A *lattice* $L$ is a free abelian group (free $\mathbb{Z}$-module) of finite rank (dimension) with a non-degenerate, symmetric bilinear form $\langle\cdot,\cdot\rangle\colon L \times L \to \mathbb{Q}$.
- The *norm* of a lattice vector $\alpha \in L$ is $\langle\alpha,\alpha\rangle/2$ and the *distance* of two lattice vectors $\alpha, \beta \in L$ is $\langle\alpha-\beta,\alpha-\beta\rangle/2$.
- The *dual lattice* $L'$ of a lattice $L$ is given by

$$L' = \{\alpha \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle\alpha,\beta\rangle \in \mathbb{Z} \text{ for all } \beta \in L\}$$

- A lattice $L$ is called *integral* if $\langle\alpha,\beta\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in L$, i.e. if $L \subseteq L'$.
- A lattice $L$ is called *even* if $\langle\alpha,\alpha\rangle \in 2\mathbb{Z}$ for all $\alpha \in L$.
- A lattice $L$ is called *positive definite* if the linear extension of $\langle\cdot,\cdot\rangle$ to $L \otimes_{\mathbb{Z}} \mathbb{R}$ is.
- A lattice $L$ is called *unimodular* if $L = L'$.

# Positive Definite, Even, Unimodular Lattices

- The dimension of a positive definite, even, unimodular lattice is in $8\mathbb{Z}_{\geq 0}$.
- Classification known up to dimension 24:

| Dimension | No. of Lattices | Lattices |
|---|---|---|
| 0 | 1 | $\{0\}$ |
| 8 | 1 | $E_8$ |
| 16 | 2 | $E_8^2$, $D_{16}^+$ |
| 24 | 24 | 24 Niemeier lattices |
| 32 | $\geq 1\,160\,000\,000$ | |

- Interesting case of *Niemeier lattices* in dimension 24 [Nie73, Ven80, CS99].

# The Niemeier Lattices

- The *roots* of an even, unimodular lattice $L$ are exactly the vectors $\alpha \in L$ of norm $\langle\alpha,\alpha\rangle/2 = 1$.
- The roots of a Niemeier lattice form a (simply-laced) *root system* $\Phi$.
- The Niemeier lattices are classified by their root systems $\Phi$: $\emptyset$, $A_1^{24}$, $A_2^{12}$, $A_3^8$, $A_4^6$, $A_5^4 D_4$, $D_4^6$, $A_6^4$, $A_7^2 D_5^2$, $A_8^3$, $A_9^2 D_6$, $D_6^4$, $A_{11} D_7 E_6$, $E_6^4$, $A_{12}^2$, $D_8^3$, $A_{15} D_9$, $A_{17} E_7$, $D_{10} E_7^2$, $D_{12}^2$, $A_{24}$, $E_8^3$, $D_{16} E_8$, $D_{24}$.
- Denote by $N(\Phi)$ the up to isomorphism unique Niemeier lattice with root system $\Phi$.
- The *Leech lattice* $\Lambda = N(\emptyset)$ is the unique Niemeier lattice without roots.

# Deep-Hole Construction of 23 Niemeier lattices

- A *hole* of a positive definite lattice $L$ is a point in $L \otimes_{\mathbb{Z}} \mathbb{R}$ where the minimal distance to any lattice vector has a local maximum. In a *deep hole* this is a global maximum.
- The *vertices* of a hole are the vectors in $L$ closest to the hole.
- The Leech lattice $\Lambda$ has 23 orbits under $\mathrm{Aut}(\Lambda)$ of deep holes with minimal distance to any lattice vector of 1 [CS99].
- The vertices $V$ of the deep holes in the Leech lattice $\Lambda$ form extended affine Dynkin diagrams by joining two vertices by
  - no edge if they have a distance of 2,
  - a simple edge if they have a distance of 3,
  - a double edge if they have a distance of 4.
- The corresponding finite Dynkin diagrams describe exactly the 23 root systems $\Phi$ of the Niemeier lattices [CS99].

# Deep-Hole Construction of 23 Niemeier lattices

| $\Phi$ | $n$ | $\|V\|$ |
|---|---|---|
| $A_1^{24}$ | 2 | 48 |
| $A_2^{12}$ | 3 | 36 |
| $A_3^8$ | 4 | 32 |
| $A_4^6$ | 5 | 30 |
| $A_5^4 D_4$ | 6 | 29 |
| $D_4^6$ | 6 | 30 |
| $A_6^4$ | 7 | 28 |
| $A_7^2 D_5^2$ | 8 | 28 |
| $A_8^3$ | 9 | 27 |
| $A_9^2 D_6$ | 10 | 27 |
| $D_6^4$ | 10 | 28 |
| $A_{11} D_7 E_6$ | 12 | 27 |

| $\Phi$ | $n$ | $\|V\|$ |
|---|---|---|
| $E_6^4$ | 12 | 28 |
| $A_{12}^2$ | 13 | 26 |
| $D_8^3$ | 14 | 27 |
| $A_{15} D_9$ | 16 | 26 |
| $A_{17} E_7$ | 18 | 26 |
| $D_{10} E_7^2$ | 18 | 27 |
| $D_{12}^2$ | 22 | 26 |
| $A_{24}$ | 25 | 25 |
| $E_8^3$ | 30 | 27 |
| $D_{16} E_8$ | 30 | 26 |
| $D_{24}$ | 46 | 25 |

Section 2

# Schellekens' List

# Definition (Vertex Operator Algebra)

- graded (by weights) $\mathbb{C}$-vector space

$$V = \bigoplus_{n=0}^{\infty} V_n \quad \text{with} \quad \dim(V_n) < \infty$$

  and vacuum $V_0 = \mathbb{C}\mathbf{1}$,
- state-field correspondence

$$Y(\cdot, z) : V \to \operatorname{End}(V)[[z, z^{-1}]],$$
$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$$

  with $v_n u = 0$ for $n \gg 0$ and $\operatorname{wt}(v_n) = \operatorname{wt}(v) - n - 1$,
- axioms: vacuum axiom, translation axiom, generalised commutativity and associativity,
- central charge $c \in \mathbb{C}$

# Examples

## Example (The Moonshine Module)

- vertex operator algebra $V^\natural$ of central charge $c = 24$,
- automorphism group $\mathrm{Aut}(V^\natural) \cong M$, Monster group,
- constructed by Frenkel, Lepowsky, Meurman [FLM88],
- needed for Borcherds' proof [Bor92] of the Moonshine conjecture

## Example (Lattice Vertex Operator Algebras [FLM88, Don93])

- $L$ positiv definite, even lattice,
- lattice vertex operator algebra $V_L$ of central charge $c = \mathrm{rk}(L)$

- The vertex operator algebras in the two examples are "nice".

# Nice Vertex Operator Algebras

Regularity assumptions on nice vertex operator algebras $V$:

- *Rationality*: Every $V$-module is completely reducible and the set $\mathrm{Irr}(V)$ of isomorphism classes of irreducible $V$-modules is finite.
- *$C_2$-cofiniteness*: The linear span $\langle\{a_{-2}b \mid a, b \in V\}\rangle$ has finite codimension in $V$.
- *Simplicity*
- *Self-duality*

Vertex operator algebras with trivial representation theory:

- *Holomorphicity*: $V$ has only one irreducible module, namely $V$ itself.

# Schellekens' List

**Proposition (Consequence of [Zhu96])**

*Let $V$ be a nice, holomorphic vertex operator algebra. Then the central charge $c$ of $V$ is in $8\mathbb{Z}_{>0}$.*

- $c = 8$: $V_{E_8}$, $c = 16$: $V_{E_8^2}, V_{D_{16}^+}$ (only lattice theories)

**Theorem ([Sch93, EMS15])**

*Let $V$ be a nice, holomorphic vertex operator algebra of central charge $c = 24$. Then the Lie algebra $V_1$ is isomorphic to one of the 71 Lie algebras on Schellekens' list ($V^\natural$, 24 lattice theories, etc. with $\operatorname{ch}_V(\tau) = j(\tau) - 744 + \dim(V_1)$).*

- $c = 32$: already more than 1 160 000 000 lattice theories

# Constructions

- Orbifold constructions give all 71 cases on Schellekens' list [FLM88, DGM90, Don93, DGM96, Lam11, LS12, LS15, Miy13, SS16, EMS15, Mö16, LS16b, LS16a, LL16]

**Theorem (Classification I)**

*There is a nice, holomorphic vertex operator algebra $V$ of central charge $c = 24$ with Lie algebra $V_1$ if and only if $V_1$ is isomorphic to one of the 71 Lie algebras on Schellekens' list.*

**Conjecture (Classification II)**

*There are up to isomorphism exactly 71 nice, holomorphic vertex operator algebras $V$ of central charge $c = 24$.*

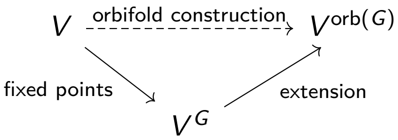
- Uniqueness essentially proved for all cases except $V^\natural$ [DM04, LS16c, KLL16, LS15, LL16, EMS17, LS17].

# Schellekens' List

| Rk. \ D. | 0 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | 108 | 120 | 132 | 144 | 156 | 168 | 192 | 216 | 240 | 264 | 288 | 300 | 312 | 336 | 360 | 384 | 408 | 456 | 552 | 624 | 744 | 1128 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $0$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | | | $C_{4,10}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 6 | | | $A_{2,6}D_{4,12}$ | $A_{6,7}$, $A_{1,2}D_{5,8}$ | $A_{2,2}F_{4,6}$ | | | | | | | | | | | | | | | | | | | | | | | | | |
| 8 | | | | $A_{1,2}A_{5,6}B_{2,3}$, $A_{4,5}^2$ | | $A_1^2D_{6,5}$, $A_1C_{5,3}G_{2,2}$ | | | | | | | | | | | | | | | | | | | | | | | | |
| 10 | | | | $A_{1,2}A_{3,4}^3$ | | $A_1^3A_{7,4}$, $A_1^2C_{3,2}D_{5,4}$ | | $A_2B_2E_{6,4}$ | | $A_3C_{7,2}$ | | | | | | | | | | | | | | | | | | | | |
| 12 | | | $A_{1,4}^{12}$ | $A_{2,3}^6$ | $A_{2,2}^4D_{4,4}$, $B_{2,2}^6$ | $A_1^3A_{5,3}D_{4,3}$ | $A_{4,2}^2C_{4,2}$, $B_{3,2}^4$ | $A_2^2A_{8,3}$ | $B_{4,2}^3$ | $A_3D_{7,3}G_2$, $E_{6,3}G_2^3$ | $A_{8,2}F_{4,2}$ | | $B_{6,2}^2$ | $A_5E_{7,3}$ | | | | | | $B_{12,2}$ | | | | | | | | | | |
| 16 | | | | $A_{1,2}^{16}$ | | $A_1^4A_{3,2}^4$ | | $A_2^2A_{5,2}^2B_2$, $B_2^4D_{4,2}^2$ | | $A_3^2D_{5,2}^2$, $A_3A_{7,2}C_3^2$ | | $A_4A_{9,2}B_3$, $B_3^2C_4D_{6,2}$, $C_4^4$ | | $A_5C_5E_{6,2}$ | $B_4^2D_{8,2}$, $B_4C_6^2$ | $A_7D_{9,2}$ | $B_5E_{7,2}F_4$, $C_8F_4^2$ | | $B_6C_{10}$ | | | | | $B_8E_{8,2}$ | | | | | | |
| 24 | | $\mathbb{C}^{24}$ | | | | $A_1^{24}$ | | $A_2^{12}$ | | $A_3^8$ | | $A_4^6$ | | $A_5^4D_4$, $D_4^6$ | $A_6^4$ | $A_7^2D_5^2$ | $A_8^3$ | $A_9^2D_6$, $D_6^4$ | | | $A_{11}D_7E_6$, $E_6^4$ | $A_{12}^2$ | $D_8^3$ | | $A_{15}D_9$ | $A_{17}E_7$, $D_{10}E_7^2$ | $D_{12}^2$ | $A_{24}$ | $E_8^3$, $D_{16}E_8$ | $D_{24}$ |

# Orbifold Construction [Mö16, EMS17]

- Let $V$ be a nice, holomorphic vertex operator algebra and $G = \langle g \rangle$ a finite, cyclic subgroup of $\mathrm{Aut}(V)$ of order $n$.
- Then $V^G$ is nice and the fusion algebra of $V^G$ is the group algebra of a central extension of $\mathbb{Z}_n$ by $\mathbb{Z}_n$.
- Obtain new holomorphic vertex operator algebras by adding $V^G$-modules corresponding to maximal isotropic subgroups:




**Example**

The Moonshine module $V^\natural$ is an orbifold of $V_\Lambda$ of order 2 [FLM88].

# Orbifolds of the Leech Lattice Vertex Operator Algebra

- Automorphisms in $\mathrm{Aut}(V_\Lambda)$ are of the form $g = \hat{\nu}\mathrm{e}^{(2\pi\mathrm{i})h_0}$ for $h \in \pi_\nu(\mathfrak{h})$ where $\mathfrak{h} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \cong (V_\Lambda)_1$.
- Search for finite-order automorphisms $g$ such that:
  - $(V_\Lambda^g)_1 \cong \pi_\nu(\mathfrak{h})$ is a Cartan subalgebra of $(V_\Lambda^{\mathrm{orb}(g)})_1$,
  - the conformal weights

$$\rho(V_\Lambda(g^i)) = \rho_{\nu^i} + \min_{\alpha \in \pi_{\nu^i}(\Lambda)+ih} \langle \alpha, \alpha \rangle / 2$$

    for $i \in \mathbb{Z}_n \setminus \{0\}$ are large, e.g. all equal to 1.
- Observations for "extremal" cases:
  - $\dim((V_\Lambda^{\mathrm{orb}(g)})_1)$ is determined by dimension formulae in [EMS17] (even when not applicable).
  - The lattice $\Lambda^{\nu,h} := \{\alpha \in \Lambda \mid \nu\alpha = \alpha, \langle \alpha, h \rangle \in \mathbb{Z}\}$ is related to the orbit lattices in [Hö17]. (Note that $V_{\Lambda^{\nu,h}} \subseteq V_\Lambda^g$.)

# Special Case: Deep Holes

- Automorphism of the form $g = \mathrm{e}^{(2\pi\mathrm{i})h_0}$ for $h \in \mathfrak{h} \cong (V_\Lambda)_1$. Orbifold must yield a Niemeier lattice vertex operator algebra.
- $V_\Lambda^g = V_{\Lambda^h}$ with $\Lambda^h = \{\alpha \in \Lambda \mid \langle \alpha, h \rangle \in \mathbb{Z}\}$.
- Let $h$ be a deep hole of $\Lambda$ with $nh \in \Lambda$. Then

$$\rho(V_\Lambda(g)) = \min_{\alpha \in \Lambda + h} \langle \alpha, \alpha \rangle / 2 = 1.$$

- Then $\Lambda^h \overset{n}{\subseteq} \Lambda = \Lambda' \overset{n}{\subseteq} (\Lambda^h)'$ and $(\Lambda^h)' = \mathrm{span}_\mathbb{Z}\{\Lambda, h\}$.
- Irreducible $V_\Lambda^g$-modules are indexed by $(\Lambda^h)'/\Lambda^h \cong \mathbb{Z}_n^2$.
- There are two (trivially intersecting) maximal isotropic subgroups

$$I_0 = \Lambda/\Lambda^h \text{ and } I = \bigcup_{i \in \mathbb{Z}_n} (\Lambda_h + ih)/\Lambda^h = N/\Lambda^h$$

  for Niemeier lattice $N(\Phi)$ from above.
- Sum of modules corresponding to $I$ is $V_\Lambda^{\mathrm{orb}(g)} = V_{N(\Phi)}$.

# Results and Open Questions

- So far, we have constructed 57 (63) cases on Schellekens' list as cyclic orbifolds of $V_\Lambda$.

**Conjecture**

*Each of the 71 vertex operator algebras on Schellekens' list is a cyclic orbifold of the Leech lattice vertex operator algebra (in a certain uniform way).*

- Try to find a uniform description of all the cases on Schellekens' list.
- Relate to the work of [Hö17].
- Fully relate the deep-hole construction of $V_{N(\Phi)}$ to that of $N(\Phi)$.

Thank you for your attention!

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