Realizing elliptic and hyperbolic Eisenstein series as theta lifts

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Holomorphic modular forms

Definition

A function $f : \mathbb{H} \to \mathbb{C}$ is called **modular form** of weight $k \in \mathbb{Z}$ if

- $f(z) = (f|_k M)(z) := (cz + d)^{-k} f(Mz)$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$,
- f is holomorphic on $\mathbb{H}$, and
- f is holomorphic at ∞.

Holomorphic modular forms

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- f is holomorphic on $\mathbb{H}$, and
- f is holomorphic at ∞.

Example: For $k \geqslant 4$ even, the classical holomorphic Eisenstein series is given by

$$E_k(z) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} 1\Big|_k M = \sum_{(c,d)=1} \frac{1}{(cz+d)^k}.$$

It is a modular form of weight k.

More holomorphic modular forms

Definition (Zagier, 1975; Bengoechea, 2013)

For $k \geqslant 2$ and $D \in \mathbb{Z}$ a discriminant, the function $f_{k,D}(z)$ is given by

$$f_{k,D}(z) = \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(z,1)^k}.$$

Here $\mathcal{Q}_D$ = set of integral binary quadratic forms of disc. D.

More holomorphic modular forms

Definition (Zagier, 1975; Bengoechea, 2013)

For $k \geqslant 2$ and $D \in \mathbb{Z}$ a discriminant, the function $f_{k,D}(z)$ is given by

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Here $\mathcal{Q}_D$ = set of integral binary quadratic forms of disc. D.

Then $f_{k,D}(z)$ is a (meromorphic) modular form of weight $2k$, which is

- a cusp form (i.e. vanishes at ∞) if $D > 0$,
- a multiple of the Eisenstein series $E_{2k}(z)$ if $D = 0$,
- a "meromorphic cusp form" (i.e. vanishes at ∞, but has poles at all CM-points of discriminant D in $\mathbb{H}$) if $D < 0$.

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The classical non-holomorphic Eisenstein series

The classical **non-holomorphic** Eisenstein series is given by

$$E_\infty(z,s) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} y^s \Big|_k M = \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

It is a non-holomorphic modular form in the following sense:

- modular of weight 0 in $z \in \mathbb{H}$
- smooth in $z \in \mathbb{H}$, satisfying

$$\Delta_{\mathrm{hyp}} E_\infty(z,s) = s(1-s) E_\infty(z,s),$$

  where $\Delta_{\mathrm{hyp}} = -y^2 (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$
- behaves 'nicely' at ∞

Hyperbolic non-holomorphic Eisenstein series

Definition (Kudla and Milson, 1979)

Given a geodesic c in $\mathbb{H}$ the associated **hyperbolic Eisenstein series** is given by

$$E_c^{\mathrm{hyp}}(z,s) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_c \setminus \mathrm{SL}_2(\mathbb{Z})} \cosh(d_{\mathrm{hyp}}(Mz,c))^{-s}$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

- modular of weight 0 in $z \in \mathbb{H}$
- smooth in $z \in \mathbb{H}$, satisfying

$$\Delta_{\mathrm{hyp}} E_{x,x'}^{\mathrm{hyp}}(z,s) = s(1-s)E_{x,x'}^{\mathrm{hyp}}(z,s) + s^2 E_{x,x'}^{\mathrm{hyp}}(z,s+2)$$

- vanishes at ∞

Elliptic non-holomorphic Eisenstein series

Definition (Jorgenson and Kramer, 2004; v. Pippich, 2005)

Given $w \in \mathbb{H}$ the associated **elliptic Eisenstein series** is given by

$$E_w^{\mathrm{ell}}(z,s) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_w \backslash \mathrm{SL}_2(\mathbb{Z})} \sinh(d_{\mathrm{hyp}}(Mz,w))^{-s}$$

for $z \in \mathbb{H}$ with $z \neq w \bmod \mathrm{SL}_2(\mathbb{Z})$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

- modular of weight 0 in z
- smooth in $z \in \mathbb{H}$, satisfying

$$\Delta_{\mathrm{hyp}} E_{x,x'}^{\mathrm{hyp}}(z,s) = s(1-s) E_{x,x'}^{\mathrm{hyp}}(z,s) - s^2 E_{x,x'}^{\mathrm{hyp}}(z,s+2)$$

- vanishes at ∞

Holomorphic and non-holomorphic Eisenstein series

The (holomorphic) functions $f_{k,D}$ generalize $E_{2k}(z)$, i.e.,

$$E_{2k}(z) = \sum_{(c,d)=1} \frac{1}{(cz+d)^{2k}} \quad \rightarrow \quad f_{k,D}(z) = \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(z,1)^k}.$$

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Analogously, we define functions $g_D(z,s)$ generalizing $E_\infty(z,s)$ via

$$E_\infty(z,s) = \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}} \quad \rightarrow \quad g_D(z,s) = \sum_{Q \in \mathcal{Q}_D} \frac{y^s}{|Q(z,1)|^s}.$$

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Question

Is there a relation between $E_c^{\mathrm{hyp}}(z,s)$ and $E_w^{\mathrm{ell}}(z,s)$, and the above function $g_D(z,s)$?

Holomorphic and non-holomorphic Eisenstein series

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Analogously, we define functions $g_D(z,s)$ generalizing $E_\infty(z,s)$ via

$$E_\infty(z,s) = \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}} \quad \rightarrow \quad g_D(z,s) = \sum_{Q \in \mathcal{Q}_D} \frac{y^s}{|Q(z,1)|^s}.$$

Lemma

We have

$$g_D(z,s) = \begin{cases} D^{-s/2} \displaystyle\sum_{Q \in \mathcal{Q}_D/\mathrm{SL}_2(\mathbb{Z})} E^{\mathrm{hyp}}_{c_Q}(z,s), & \text{if } D > 0, \\ 2\zeta(s)\ E_\infty(z,s), & \text{if } D = 0, \\ |D|^{-s/2} \displaystyle\sum_{Q \in \mathcal{Q}_D/\mathrm{SL}_2(\mathbb{Z})} E^{\mathrm{ell}}_{w_Q}(z,s), & \text{if } D < 0. \end{cases}$$

	holomorphic world	**non-holomorphic world**
discriminant D	$f_{k,D}(z) = \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(z,1)^k}$	$g_D(z,s) = \sum_{Q \in \mathcal{Q}_D} \frac{y^s}{\lvert Q(z,1)\rvert^s}$
$D > 0$	Zagier's cusp forms	$\sum_{Q \in \mathcal{Q}_D/\,\mathrm{SL}_2(\mathbb{Z})} E^{\mathrm{hyp}}_{c_Q}(z,s)$ averaged hyperbolic Eisenstein series
$D = 0$	$E_{2k}(z)$ Eisenstein series	$E_\infty(z,s)$ (averaged) parabolic Eisenstein series
$D < 0$	Bengoecheas's meromorphic cusp forms	$\sum_{Q \in \mathcal{Q}_D/\,\mathrm{SL}_2(\mathbb{Z})} E^{\mathrm{ell}}_{w_Q}(z,s)$ averaged elliptic Eisenstein series

A side note on generating series

Theorem (Kohnen and Zagier, 1981)

For $k \geqslant 2$ and $z \in \mathbb{H}$ the function

$$\Omega_k(\tau, z) = \sum_{D>0} D^{k-1/2} f_{k,D}(z) e(D\tau)$$

is a modular form of weight $k + 1/2$ in τ.

A side note on generating series

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Question

Is there a non-holomorphic analog for the functions $g_D(z, s)$?

A side note on generating series

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For $k \geqslant 2$ and $z \in \mathbb{H}$ the function

$$\Omega_k(\tau, z) = \sum_{D>0} D^{k-1/2} f_{k,D}(z) e(D\tau)$$

is a modular form of weight $k + 1/2$ in τ.

Question

Is there a non-holomorphic analog for the functions $g_D(z, s)$?

- In the variable τ, the generating series should neither be holomorphic, nor an eigenfunction of Δ_{hyp}.
- Replace $e(D\tau)$ by something appropiate, for example by $v^s \mathcal{M}_{k,s}(4\pi D v) e(Du)$?

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General theta lifts

Roughly, a theta lift is a map of the form

$$f(\tau) \quad \mapsto \quad \Phi(z,f) := \int f(\tau)\cdot\Theta(\tau,z),$$

where:

- $f(\tau) =$ modular object of type A
- $\Phi(z,f) =$ modular object of type B
- $\Theta(\tau,z) =$ appropriate theta kernel

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The Shintani lift

For $k \geqslant 2$ let $\Theta_k^{\mathsf{Sh}}(\tau, z)$ be the **Shintani theta function**, which is

- modular of weight $k + 1/2$ in τ, and
- modular of weight $2k$ in z.

For a certain lattice (L, q) of signature $(2, 1)$ we can write

$$\Theta_k^{\mathsf{Sh}}(\tau, z) = v^{1/2} y^{-2k} \sum_{X \in L} Q_X(\overline{z}, 1)^k e\Big(uq(X) + ivq_z(X)\Big).$$

Definition

Let f be a modular form of weight $k + 1/2$. The **Shintani lift** of f given by

$$\Phi_k^{\mathsf{Sh}}(z; f) = \int_{\mathsf{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathsf{reg}} f(\tau) \overline{\Theta_k^{\mathsf{Sh}}(\tau, z)} v^k \, \frac{du\, dv}{v^2}$$

is a modular form of weight $2k$.

Lifting holomorphic Poincaré series

Let $P_{k,D}(\tau)$ be the (holomorphic) Poincaré series of weight k, i.e.,

$$P_{k,D}(\tau) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} e(D\tau)\Big|_k M.$$

Lifting holomorphic Poincaré series

Let $P_{k,D}(\tau)$ be the (holomorphic) Poincaré series of weight k, i.e.,

$$P_{k,D}(\tau) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} e(D\tau)\Big|_k M.$$

Theorem

Let $k \geqslant 2$ and let $D \in \mathbb{Z}$. The Shintani lift of $P_{k+1/2,D}(\tau)$ is given by

$$\Phi_k^{\mathrm{Sh}}(z; P_{k+1/2,D}) = \frac{\Gamma(k)}{\pi^k} f_{k,D}(z).$$

	holomorphic world	**non-holomorphic world**
discriminant D	$f_{k,D}(z) = \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(z,1)^k}$	$g_D(z,s) = \sum_{Q \in \mathcal{Q}_D} \frac{y^s}{\lvert Q(z,1) \rvert^s}$
$D > 0$	Zagier's cusp forms	$\sum_{Q \in \mathcal{Q}_D / \mathrm{SL}_2(\mathbb{Z})} E^{\mathrm{hyp}}_{c_Q}(z,s)$ averaged hyperbolic Eisenstein series
$D = 0$	$E_{2k}(z)$ Eisenstein series	$E_\infty(z,s)$ (averaged) parabolic Eisenstein series
$D < 0$	Bengoecheas's meromorphic cusp forms	$\sum_{Q \in \mathcal{Q}_D / \mathrm{SL}_2(\mathbb{Z})} E^{\mathrm{ell}}_{w_Q}(z,s)$ averaged elliptic Eisenstein series
Realization as theta lift	Shintani lift of holomorphic Poincaré series $P_{k+1/2,D}(\tau)$	**?**

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Borcherds' regularized theta lift of signature $(2, 1)$

Let $\Theta_L(\tau, z)$ be the classical **Siegel theta function**, which is

- modular of weight $1/2$ in τ, and
- modular of weight 0 in z.

Using the lattice (L, q) of signature $(2, 1)$ from before we can write

$$\Theta_L(\tau, z) = v^{1/2} \sum_{X \in L} e\Big(uq(X) + ivq_z(X)\Big).$$

Definition

Let f be modular of weight $1/2$. The **Borcherds lift** of f given by

$$\Phi^{\mathrm{Bo}}(z; f) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} f(\tau)\overline{\Theta_L(\tau, z)} v^{1/2} \frac{du\,dv}{v^2}$$

is modular of weight 0.

Lifting non-holomorphic Poincaré series

Let $U_D(\tau)$ be Selberg's (non-holomorphic) Poincaré series of weight $1/2$, i.e.,

$$U_D(\tau, s) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} v^s e(D\tau) \Big|_{1/2} M.$$

Lifting non-holomorphic Poincaré series

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$$U_D(\tau, s) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} v^s e(D\tau) \Big|_{1/2} M.$$

Theorem (Schwagenscheidt, V. and v. Pippich, 2017)

Let $D \in \mathbb{Z}$ be a discriminant. Then

$$\Phi^{\mathrm{Bo}}(z; U_D(\,\cdot\,, s)) = \frac{\Gamma(s)}{\pi^s}\, g_D(z, 2s).$$

Lifting non-holomorphic Poincaré series

Let $U_D(\tau)$ be Selberg's (non-holomorphic) Poincaré series of weight $1/2$, i.e.,

$$U_D(\tau, s) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} v^s e(D\tau) \Big|_{1/2} M.$$

Theorem (Schwagenscheidt, V. and v. Pippich, 2017)

Let $D \in \mathbb{Z}$ be a discriminant. Then

$$\Phi^{\mathrm{Bo}}(z; U_D(\,\cdot\,, s)) = \begin{cases} \dfrac{\Gamma(s)}{(D\pi)^s} \displaystyle\sum_{Q \in \mathcal{Q}_D / \mathrm{SL}_2(\mathbb{Z})} E^{\mathrm{hyp}}_{c_Q}(z, 2s), & \text{if } D > 0, \\ \dfrac{2\zeta(2s)\Gamma(s)}{\pi^s} \, E_\infty(z, 2s), & \text{if } D = 0, \\ \dfrac{\Gamma(s)}{(|D|\pi)^s} \displaystyle\sum_{Q \in \mathcal{Q}_D / \mathrm{SL}_2(\mathbb{Z})} E^{\mathrm{ell}}_{w_Q}(z, 2s), & \text{if } D < 0. \end{cases}$$

	holomorphic world	**non-holomorphic world**
discriminant D	$f_{k,D}(z) = \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(z,1)^k}$	$g_D(z,s) = \sum_{Q \in \mathcal{Q}_D} \frac{y^s}{\lvert Q(z,1)\rvert^s}$
$D > 0$	Zagier's cusp forms	$\sum_{Q \in \mathcal{Q}_D / \mathrm{SL}_2(\mathbb{Z})} E^{\mathrm{hyp}}_{c_Q}(z,s)$ averaged hyperbolic Eisenstein series
$D = 0$	$E_{2k}(z)$ Eisenstein series	$E_\infty(z,s)$ (averaged) parabolic Eisenstein series
$D < 0$	Bengoecheas's meromorphic cusp forms	$\sum_{Q \in \mathcal{Q}_D / \mathrm{SL}_2(\mathbb{Z})} E^{\mathrm{ell}}_{w_Q}(z,s)$ averaged elliptic Eisenstein series
Realization as theta lift	Shintani lift of holomorphic Poincaré series $P_{k+1/2,D}(\tau)$	Borcherds lift of Selberg's Poincaré series $U_D(\tau,s)$

Eisenstein series associated to quadratic forms

For a given discriminant D the averaged functions

$$f_{k,D}(z) = \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(z,1)^k} \quad \text{and} \quad g_D(z,s) = \sum_{Q \in \mathcal{Q}_D} \frac{y^s}{|Q(z,1)|^s}$$

can be decomposed into sums of

$$f_{k,[Q_0]}(z) = \sum_{Q \in [Q_0]} \frac{1}{Q(z,1)^k} \quad \text{and} \quad g_{[Q_0]}(z,s) = \sum_{Q \in [Q_0]} \frac{y^s}{|Q(z,1)|^s}.$$

Here $Q_0 =$ a quadratic form of discriminant D, and

$$g_{[Q_0]}(z,s) \doteq \begin{cases} E^{\text{hyp}}_{c_{Q_0}}(z,s), & \text{if } D > 0, \\ E_{p_{Q_0}}(z,s), & \text{if } D = 0, \\ E^{\text{ell}}_{w_{Q_0}}(z,s), & \text{if } D < 0. \end{cases}$$

	holomorphic world	**non-holomorphic world**
quadratic form Q_0	$f_{k,[Q_0]}(z) = \sum_{Q \in [Q_0]} \frac{1}{Q(z,1)^k}$	$g_{[Q_0]}(z,s) = \sum_{Q \in [Q_0]} \frac{y^s}{\lvert Q(z,1) \rvert^s}$
$D > 0$	hyperbolic Poincaré series	$E^{\mathrm{hyp}}_{c_{Q_0}}(z,s)$ hyperbolic Eisenstein series
$D = 0$	holomorphic Eisenstein series	$E_{p_{Q_0}}(z,s)$ parabolic Eisenstein series
$D < 0$	elliptic Poincaré series	$E^{\mathrm{ell}}_{w_{Q_0}}(z,s)$ elliptic Eisenstein series
Realization as theta lift	**?**	**?**

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Borchers' regularized theta lift of signature $(2, 2)$

Let $\Theta_L(\tau, z, z')$ be the **Siegel theta function** associated to a certain lattice of signature $(2, 2)$, which is

- modular of weight 0 in τ, z, z'.

One can write

$$\Theta_L(\tau, z, z') = v \sum_{X \in L} e\Big(uq(X) + ivq_{z,z'}(X)\Big).$$

Definition

Let f be modular of weight 0. The **Borcherds lift** of f given by

$$\Phi^{\mathrm{Bo}}(z, z'; f) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} f(\tau)\overline{\Theta_L(\tau, z, z')}\ \frac{du\,dv}{v^2}$$

is modular of weight 0 in z and z'.

Lifting non-holomorphic Poincaré series in signature $(2,2)$

Instead of Selberg's Poincaré series

$$U_m(\tau,s) = \sum_{M\in \mathrm{SL}_2(\mathbb{Z})_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} v^s e(m\tau)\Big|_0 M,$$

we now consider the (non-holomorphic) Poincaré series

$$F_m(\tau,s) = \sum_{M\in \mathrm{SL}_2(\mathbb{Z})_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} v^s e(mu)\Big|_0 M.$$

Lifting non-holomorphic Poincaré series in signature $(2,2)$

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we now consider the (non-holomorphic) Poincaré series

$$F_m(\tau, s) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} v^s e(mu) \Big|_0 M.$$

Theorem (V.)

Let $m = -1$. Then

$$\Phi^{\mathrm{Bo}}(z, z'; F_{-1}(\,\cdot\,, s)) = \frac{\Gamma(s)}{(2\pi)^s} K_s^{\mathrm{hyp}}(z, z'),$$

where $K_s^{\mathrm{hyp}}(z, z')$ is the **hyperbolic kernel function**

$$K_s^{\mathrm{hyp}}(z, z') = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})} \cosh(d_{\mathrm{hyp}}(Mz, z'))^{-s}.$$

Identities for the hyperbolic kernel function

Question

Can we relate $K_s^{\mathrm{hyp}}(z, z')$ *to* $E_c^{\mathrm{hyp}}(z, s)$ *and* $E_w^{\mathrm{ell}}(z, s)$*?*

Identities for the hyperbolic kernel function

Question

Can we relate $K_s^{\mathrm{hyp}}(z,z')$ to $E_c^{\mathrm{hyp}}(z,s)$ and $E_w^{\mathrm{ell}}(z,s)$?

Lemma (Jorgenson, Smajlovic and v. Pippich, 2016)

1. *Let c be a closed geodesic in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Then*

$$E_c^{\mathrm{hyp}}(z,s) = \frac{\Gamma(s)}{2^s\Gamma(s/2)^2} \int\limits_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c} K_s^{\mathrm{hyp}}(z,w)\,\frac{dw}{Q_c(w,1)}.$$

2. *Let $w \in \mathbb{H}$. Then*

$$E_w^{\mathrm{ell}}(z,s) = \frac{1}{|\,\mathrm{SL}_2(\mathbb{Z})_w|} \sum_{n=0}^{\infty} \frac{(s/2)_n}{n!} K_{s+2n}^{\mathrm{hyp}}(z,w).$$

Realizing distinguished hyperbolic Eisenstein series

Let c be a closed geodesic in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Then

$$E_c^{\mathrm{hyp}}(z,s) \doteq \int_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c} K_s^{\mathrm{hyp}}(z,w)\,\frac{dw}{Q_c(w,1)}$$

Realizing distinguished hyperbolic Eisenstein series

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$$\doteq \int_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c} \Phi^{\mathrm{Bo}}(z,w;F_{-1}(\,\cdot\,,s))\,\frac{dw}{Q_c(w,1)}$$

Realizing distinguished hyperbolic Eisenstein series

Let c be a closed geodesic in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Then

$$\begin{aligned} E_c^{\mathrm{hyp}}(z,s) &\doteq \int_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c} K_s^{\mathrm{hyp}}(z,w)\,\frac{dw}{Q_c(w,1)} \\ &\doteq \int_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c} \Phi^{\mathrm{Bo}}(z,w;F_{-1}(\,\cdot\,,s))\,\frac{dw}{Q_c(w,1)} \\ &\doteq \int_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} F_{-1}(\tau,s)\overline{\Theta_L(\tau,z,w)}\,\frac{du\,dv}{v^2}\,\frac{dw}{Q_c(w,1)} \end{aligned}$$

Realizing distinguished hyperbolic Eisenstein series

Let c be a closed geodesic in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Then

$$\begin{aligned}
E_c^{\mathrm{hyp}}(z,s) &\doteq \int_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c} K_s^{\mathrm{hyp}}(z,w)\,\frac{dw}{Q_c(w,1)} \\
&\doteq \int_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c} \Phi^{\mathrm{Bo}}(z,w;F_{-1}(\,\cdot\,,s))\,\frac{dw}{Q_c(w,1)} \\
&\doteq \int_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c}\ \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} F_{-1}(\tau,s)\overline{\Theta_L(\tau,z,w)}\,\frac{du\,dv}{v^2}\,\frac{dw}{Q_c(w,1)} \\
&\doteq \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} F_{-1}(\tau,s) \int_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c} \overline{\Theta_L(\tau,z,w)}\,\frac{dw}{Q_c(w,1)}\,\frac{du\,dv}{v^2}
\end{aligned}$$

Realizing distinguished hyperbolic Eisenstein series

For a closed geodesic c in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ we define

$$\Theta_{L,c}(\tau,z) = \int_{\mathrm{SL}_2(\mathbb{Z})_c\backslash c} \overline{\Theta_L(\tau,z,w)}\,\frac{dw}{Q_c(w,1)}.$$

Corollary

The Eisenstein series $E_c^{\mathrm{hyp}}(z,s)$ can be realized as the integral

$$E_c^{\mathrm{hyp}}(z,s) = \frac{\pi^s}{\Gamma(s/2)^2}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} F_{-1}(\tau,s)\,\Theta_{L,c}(\tau,z)\,\frac{du\,dv}{v^2}.$$

Here

$$F_m(\tau,s) = \sum_{M\in\mathrm{SL}_2(\mathbb{Z})_\infty\backslash\,\mathrm{SL}_2(\mathbb{Z})} v^s e(mu)\Big|_0 M.$$

Realizing distinguished elliptic Eisenstein series

Let $w \in \mathbb{H}$. Then

$$E_w^{\mathrm{ell}}(z, 2s) \doteq \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} K_{2s+2n}^{\mathrm{hyp}}(z, w)$$

Realizing distinguished elliptic Eisenstein series

Let $w \in \mathbb{H}$. Then

$$E_w^{\text{ell}}(z, 2s) \doteq \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} K_{2s+2n}^{\text{hyp}}(z, w)$$

$$\doteq \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} \frac{(2\pi)^{2s+2n}}{\Gamma(2s+2n)} \Phi^{\text{Bo}}(z, w; F_{-1}(\,\cdot\,, 2s+2n))$$

Realizing distinguished elliptic Eisenstein series

Let $w \in \mathbb{H}$. Then

$$E_w^{\mathrm{ell}}(z, 2s) \doteq \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} K_{2s+2n}^{\mathrm{hyp}}(z, w)$$

$$\doteq \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} \frac{(2\pi)^{2s+2n}}{\Gamma(2s+2n)} \int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} F_{-1}(\tau, 2s+2n)\overline{\Theta_L(\tau, z, w)}\, \frac{du\, dv}{v^2}$$

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$$\doteq \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} \frac{(2\pi)^{2s+2n}}{\Gamma(2s+2n)}$$

$$\times \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \left(\sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} v^{2s+2n} e(-u) \Big|_0 M \right) \overline{\Theta_L(\tau, z, w)} \frac{du\, dv}{v^2}$$

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$$\times \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \left(\sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} v^{2s+2n} e(-u) \Big|_0 M \right) \overline{\Theta_L(\tau, z, w)} \frac{du\, dv}{v^2}$$

$$\doteq \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \left[\sum_{M} \left(\sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} \frac{(2\pi v)^{2s+2n}}{\Gamma(2s+2n)} \right) e(-u) \Big|_0 M \right]$$

$$\times \overline{\Theta_L(\tau, z, w)} \frac{du\, dv}{v^2}$$

Realizing distinguished elliptic Eisenstein series

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$$E_w^{\text{ell}}(z, 2s) \doteq \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} K_{2s+2n}^{\text{hyp}}(z, w)$$

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$$\times \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \left(\sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} v^{2s+2n} e(-u) \Big|_0 M \right) \overline{\Theta_L(\tau, z, w)} \, \frac{du\, dv}{v^2}$$

$$\doteq \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \left[\sum_M v^s \mathcal{M}_{0,s}(4\pi v) e(-u) \Big|_0 M \right] \overline{\Theta_L(\tau, z, w)} \, \frac{du\, dv}{v^2}$$

Realizing distinguished elliptic Eisenstein series

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$$\times \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \left(\sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} v^{2s+2n} e(-u) \Big|_0 M \right) \overline{\Theta_L(\tau, z, w)} \frac{du\, dv}{v^2}$$

$$\doteq \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \mathcal{F}_{-1}(\tau, s) \overline{\Theta_L(\tau, z, w)} \frac{du\, dv}{v^2}$$

where

$$\mathcal{F}_m(\tau, s) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} v^s \mathcal{M}_{0,s}(4\pi|m|v) e(mu) \Big|_0 M.$$

Realizing distinguished elliptic Eisenstein series

Corollary

The Eisenstein series $E_w^{\mathrm{ell}}(z,s)$ can be realized as the theta lift

$$E_w^{\mathrm{ell}}(z,2s) = \frac{1}{|\,\mathrm{SL}_2(\mathbb{Z})_w|}\,\frac{\pi^s}{\Gamma(2s)}\,\Phi^{\mathrm{Bo}}(z,w;\mathcal{F}_{-1}(\,\cdot\,,s)).$$

Here

$$\mathcal{F}_m(\tau,s) = \sum_{M\in\mathrm{SL}_2(\mathbb{Z})_\infty\backslash\,\mathrm{SL}_2(\mathbb{Z})} v^s\mathcal{M}_{0,s}(4\pi|m|v)e(mu)\Big|_0 M.$$

	holomorphic world	**non-holomorphic world**
quadratic form Q_0	$f_{k,[Q_0]}(z) = \sum_{Q\in[Q_0]} \frac{1}{Q(z,1)^k}$	$g_{[Q_0]}(z,s) = \sum_{Q\in[Q_0]} \frac{y^s}{\lvert Q(z,1)\rvert^s}$
$D > 0$	?	$E^{\mathrm{hyp}}_{c_{Q_0}}(z,s) \doteq \int F_{-1}(\tau,s)\Theta_{L,c_{Q_0}}(\tau,z)$
$D = 0$	?	?
$D < 0$	?	$E^{\mathrm{ell}}_{w_{Q_0}}(z,2s) \doteq \Phi^{\mathrm{Bo}}(z,w_{Q_0};\mathcal{F}_{-1}(\,\cdot\,,s))$

Thank you!