The Covering Radius

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Introduction

Proposition Wolfart, Wüstholz 1985

Let X be a projective, smooth algebraic curve of genus $g > 1$ that is defined over $\overline{\mathbb{Q}}$ and $x \in X$ an algebraic, $\overline{\mathbb{Q}}$-rational point. Then there is a holomorphic universal cover

$$\varphi : U_r = \{z \in \mathbb{C} \mid |z| < r\} \to X \quad \text{with} \quad \varphi(0) = x$$

and algebraic tangent map $\varphi'(0)$. The radius r is well defined in $\mathbb{R}^\times / \mathbb{R} \cap \overline{\mathbb{Q}}^\times$.

Definition

The radius of convergence $r = r(X, x)$ of the universal cover, that is defined up to algebraic multiples, is called *covering radius* of X in x.

Proposition Wolfart, Wüstholz 1985
Let Γ be the group of deck transformations of $U_1 \to X, x \mapsto 0$.
Up to algebraic multiples the covering radius $r(X,x)$ coincides with $f'(0)$ for every $\Gamma-$automorphic form that is holomorphic in 0 and fulfils $f'(0) \neq 0$.

All calculation can be made with power series - usually as solutions of differential equations

Triangle Groups

Theorem Wolfart, Wüstholz 1985
Let Π be commensurable to a triangle group Δ and $P \in \mathbb{H}$ an elliptic fixed point of Δ, then the covering radius $r(X, P)$ of $X = \Pi\backslash\mathbb{H}$ in P is transcendent.

Theorem Wolfart 1983
Let P be a parabolic fixed point of the triangle group $\Delta = \Delta(\infty, q, p)$. The covering radius $r(X, P)$ of $X = \Delta\backslash\mathbb{H}$ in P is algebraic if and only if Δ is arithmetic defined. (only 9 exists)

Proof uses solutions of the Schwarz differential operator (hypergeometric functions and Schwarz triangle maps) and Beta values that appear as periods.

Example j-invariant for $\mathrm{SL}_2(\mathbb{Z})$ at $i\infty$

$$j(\tau) = \frac{(12g_2(\tau))^3}{\Delta(\tau)} = \frac{(12g_2(\tau))^3}{g_2(\tau)^3 - 27g_3(\tau)^2}$$

Expanded in the variable $q = \exp(2\pi i\tau)$

$$j(q) = \frac{1}{q} + 744 + 196884q + \cdots .$$

has rational coefficients a pole at $q = 0$ and convergence radius 1.
Therefore the covering radius $r(\overline{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}, i\infty)$ is algebraic.

Periods

Definition

A meromorphic differential on a variety A is called *of the first kind*, if it is holomorphic on A. If it has poles and all residues vanish, then it's called *of the second kind*.

Definition

For a cycle $0 \neq \gamma \in H_1(A, \mathbb{Z})$ and a differential ω of the first resp. second kind, $\int_\gamma \omega$ is called *period of the first* resp. *second kind*.

Periods

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Theorem Wolfart, Wüstholz 1985; Cohen 2004
Let A be an abelian variety isogenous over $\overline{\mathbb{Q}}$ to the direct product

$$A \simeq A_1^{d_1} \times \cdots \times A_m^{d_m}$$

of simple, pairwise non-isogenous abelian varieties A_ν defined over $\overline{\mathbb{Q}}$, with A_ν of dimension n_ν. Then the $\overline{\mathbb{Q}}$-vector space V_A generated by $1, 2\pi i$ together with all periods of differentials, defined over $\overline{\mathbb{Q}}$, of the first and the second kind on A, has dimension

$$\dim_{\overline{\mathbb{Q}}} V_A = 2 + 4 \sum_{\nu_1}^{m} \frac{n_\nu^2}{\dim_{\mathbb{Q}} \mathrm{End}_0 A_\nu}.$$

Picard Fuchs Equations

Picard Fuchs equations

Differential equation associated to a family of curves together with a holomorphic differential.

- X_t family of curves parametrised by $t \in \mathbb{C}$, $\omega_t \in H^1(X_t, \mathbb{C})$

For every cycle $\gamma \in H_1(X_t, \mathbb{C})$ there is a period map

$$p : \mathbb{C} \to \mathbb{C}, t \mapsto \int_\gamma \omega_t$$

Dependence in cohomology gives a relation between different derivatives $\omega_t^{(k)} := \frac{\partial^k}{\partial t^k}\omega_t$. For example a family of elliptic curves E_t

$$A\omega_t'' + B\omega_t' + C\omega_t = 0 \quad \text{in} \quad H^1_{dR}(E_t)$$

Picard Fuchs equation

For every cycle the period map fulfils a differentail equation - the *Picard Fuchs equation*

$$Ap'' + Bp' + Cp = 0 \quad \text{in} \quad \mathbb{C}$$

- A, B, C are rational functions in t
- A family of genus g curves gives an equation of order $2g$
- Splits in certain cases in equations of lower order (e.g. for Teichmüller curves)

Teichmüller Curves

Teichmüller curves are curves in $\mathcal{M}_g$, that arise as $\mathrm{SL}_2(\mathbb{R})$-orbit of flat surfaces. The stabilizer is called *Veech group* - the uniformizing group of the curve.

Proposition Möller, Zagier 2016

Suppose that $W = \mathbb{H}/\Gamma$ is a Teichmüller curve in $\mathcal{M}_2$ and let L be the rank two picard Fuchs differention operator. Then there is a rank-one submodule of solutions of L consisting of holomorphic modular forms (with respect to Γ).

Suppose that $W_D = \mathbb{H}/\Gamma_D$ is a Teichmüller curve in $\mathcal{M}_2$ with discriminant D and genus $g(W_D) = 0$.

Theorem 1

Then the covering radius in a cusp (parabolic fixed point of Γ_D) is transcendent.

Theorem 2

If there are two elliptic fixed points P, P_1 of Γ_D, then the covering radius $r(W_D, P)$ in the elliptic fixed point P is transcendent.

Proof of Theorem 1 (sketch)

Find two independent modular forms f, g of the same weight on Γ_D, such that $Div(g) \subset \partial U_1$. Then f/g is an hauptmodul and by the choice of a suitable variable, the radius of convergence is the covering radius.

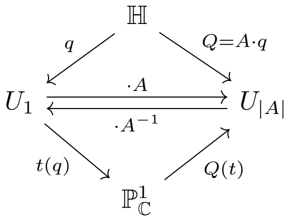
Theorem Möller, Zagier 2016
The space of all (twisted) modular forms of all (bi-) weights for the uniformizing group Γ_D of W_D has a basis of forms with Fourier expansions

$$\sum_{n\geq 0} a_n Q^n \quad \text{with} \quad a_n \in \overline{\mathbb{Q}} \quad \text{and} \quad Q = A\exp(2\pi i\tau/\alpha)$$

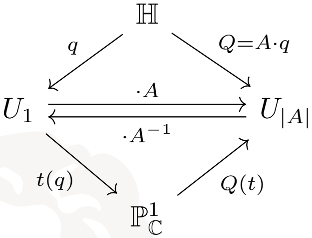
where A and the radius of convergence $|A|$ are transcendental of Gelfond-Schneider type.

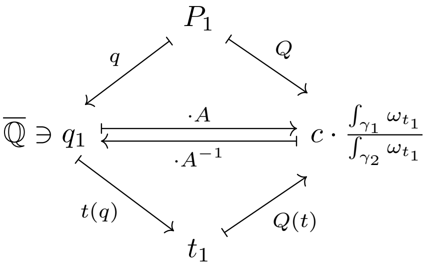
Proof of Theorem 2

- Two elliptic fixed points P_0 and P_1.
- $q = (\tau - P_0)/(\tau - \overline{P_0})$
- t is an hauptmodul i.e. an isomorphism between $\overline{\mathbb{H}/\Gamma}$ and $\mathbb{P}^1_{\mathbb{C}}$.
- $Q(t)$ is given by the quotient of solutions of the picard fuchs eq.
- $t(Q)$ and therefore $Q(t)$ have algebraic coefficients



Teichmüller Curves





- $Jac(X_{t_1})$ has CM and is isogenous to $E \times E$
- $\frac{\int_{\gamma_1} \omega_{t_1}}{\int_{\gamma_2} \omega_{t_1}} \in \overline{\mathbb{Q}}$
- $c \in \overline{\mathbb{Q}} \iff A \in \overline{\mathbb{Q}}$

Calculating c

Choose two independent cycles $\gamma_1, \gamma_2 \in H_1(X_t, \mathbb{Z})$ and use the Picard Fuchs equation to get the power series developments

$$f_1(t) = \int_{\gamma_1} \omega_t = \sum_{n \geq 0} a_n t^n \quad \text{and} \quad f_2(t) = \int_{\gamma_2} \omega_t = \sum_{n \geq 0} b_n t^n$$

from

$$a_0 = \int_{\gamma_1} \omega_0, \qquad a_1 = \int_{\gamma_1} \partial_t|_{t=0} \omega_t$$

$$b_0 = \int_{\gamma_2} \omega_0, \qquad b_1 = \int_{\gamma_2} \partial_t|_{t=0} \omega_t$$

Calculating c

- $Jac(X_{t_0})$ has CM and is isogenous to $E \times E$, therefore

$$\frac{a_0}{b_0} = \frac{\int_{\gamma_1} \omega_0}{\int_{\gamma_2} \omega_0} \in \overline{\mathbb{Q}}$$

- redefine γ_1 as

$$\gamma_1 := \gamma_1 - \frac{b_0}{a_0}\gamma_2 \in H_1(X_t, \overline{\mathbb{Q}}) := H_1(X_t, \mathbb{Z}) \otimes \overline{\mathbb{Q}}$$

- the quotient f_1/f_2 has the folliwing power series development

$$\frac{\int_{\gamma_1} \omega_t}{\int_{\gamma_2} \omega_t} = \sum_{n \geq 0} c_n t^n = 0 + \frac{a_1}{b_0} \cdot t + \cdots$$

- with $c_1 = \frac{a_1}{b_0} = \frac{\text{per. second kind}}{\text{per. first kind}} \notin \overline{\mathbb{Q}}$

Calculating c

- Define

$$Q(t) = c_1^{-1} \frac{\int_{\gamma_1} \omega_t}{\int_{\gamma_2} \omega_t} = c_1^{-1} \sum_{n \geq 0} c_n t^n = 0 + t + \cdots$$

- inverting power series as function yields

$$t(Q) = 0 + Q + \cdots$$

  with convergence radius $|A| \sim |c_1^{-1}| \in \mathbb{R} \backslash (\overline{\mathbb{Q}} \cap \mathbb{R})$

- all together the covering radius in an elliptic fixed point is transcendent