Generalised Moonshine

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→ not my work

Reminder: Monstrous Moonshine

- (1978) John McKay: $j(\tau)-744 = \frac{1}{q} + 0 + \underbrace{196884}_{=196883+1} q + \underbrace{21493760}_{=21296876+196883+1} q^2 + \dots$

  ↳ The coeff. of the j-function are linear combinations of the dims of the irreps of the Monster group $\mathbb{M}$ with "small" non-negative coefficients.

  Conj: There is a naturally occurring infinite-dim. graded rep. $V^\natural$ of $\mathbb{M}$ whose graded dimension (or character) is the q-expansion of the j-function

  $V^\natural = \bigoplus_{n=0}^{\infty} V_n^\natural$

- John Thompson: $j(\tau)-744 = q^{-1}\sum_{n=0}^{\infty} \dim(V_n^\natural) q^n = q^{-1}\sum_{n=0}^{\infty} \mathrm{tr}_{V_n^\natural}(id)\, q^n$

  ↳ graded trace of the identity

  Suggestion: Also interesting to look at

  $$T_g(q) := q^{-1}\sum_{n=0}^{\infty} \mathrm{tr}_{V_n^\natural}(g)\, q^n \quad \text{for any } g \in \mathbb{M}$$

  acts on module $V^\natural$

  ⇝ McKay-Thompson series

- John Conway, Simon Norton (1979): massive computation

  ⇝ evidence that there is an inf.-dim. graded rep. $V^\natural = \bigoplus_{n=0}^{\infty} V_n^\natural$ of $\mathbb{M}$ such that the graded trace $T_g(q)$ of any element is the Fourier expansion of a holomorphic function on $\mathbb{H}$ that is moreover a Hauptmodul.

→ That is, for each $g \in \mathbb{M}$, there is a group $\Gamma_g \subset SL_2(\mathbb{R})$ under which $T_g(\tau)$ is invariant, such that the quotient curve $X_g = \mathbb{H}^*/\Gamma_g$ has genus 0 and T_g generates the field of meromorphic functions on this sphere.

→ Conway-Norton conjecture: ex. of inf. dim. graded rep. of $\mathbb{M}$ with graded traces those on their list

- Igor Frenkel, James Lepowsky, Arne Meurman (1988):

Constructed a VOA $V^\natural$ with $\mathrm{Aut}(V^\natural) \cong \mathbb{M}$,

$$ch_{V^\natural}(\tau) = q^{-1} \sum_{n=0}^{\infty} \dim(V_n^\natural) q^n = j(\tau) \overset{-744}{=} T_{id}(\tau)$$

- Borcherds (1992): Proof of the Conway-Norton conjecture (→ Fields medal 1998)

Generalised Moonshine:

- Conway, Norton (1979): suggest that Moonshine is not limited to the Monster $\mathbb{M}$ but similar phenomena occur for other large finite groups
- Marissa Queen (1981): computational evidence that one can construct the expansions of many Hauptmoduln from simple combinations of dimensions of sporadic groups (other than $\mathbb{M}$)

<u>Example</u>: • Baby Monster sporadic group $\mathbb{B}$ (second largest of the 26 sporadic groups) has irreps $\underline{1}$, $\underline{4371}$, $\underline{96255}$, ...

• $$\frac{\Delta(\tau)}{\Delta(2\tau)} + \frac{\Delta(2\tau)}{\Delta(\tau)} + 24 = q^{-1} + 4372\,q + 96256\,q^2 + \dots$$

is a Hauptmodul for $\Gamma_0(2)^+$ (normaliser of $\Gamma_0(2)$ in $SL_2(\mathbb{R})$)

$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod n \right\}$

• In fact: for Moonshine-like behaviour: need to pass to a central extension $2.\mathbb{B}$ (double cover)

Similar behaviour appears for a triple cover $3.Fi_{24}$ of the sporadic group Fi_{24}.

<u>Observation:</u> Both $2.\mathbb{B}$ and $3.Fi_{24}$ are centralisers of elements of the Monster.

<u>Conjecture</u> (Norton 1987, revised 2001)

There exists a rule that assigns to each element g of the Monster simple group $\mathbb{M}$ a graded projective representation $V(g) = \bigoplus_{n \in \mathbb{Q}} V(g)_n$ of the centraliser $C_{\mathbb{M}}(g)$, and to each pair (g,h) of commuting elements of $\mathbb{M}$ a holomorphic function $Z(g,h,\tau)$ on $\mathbb{H}$ satisfying the following conditions:

(1) There is some lift $\tilde{h}$ of h to a linear transformation on $V(g)$ such that

$$Z(g,h,\tau) = \sum_{n\in\mathbb{Q}} \mathrm{tr}_{V(g)_n}(\tilde{h})\, q^{n-1}$$

(2) $Z(g,h,\tau)$ is invariant (up to constant multiplication) under simultaneous conjugation of the pair (g,h) in $\mathbb{M}$.

(3) $Z(g,h,\tau)$ is either constant or a Hauptmodul for some genus 0 congruence group

(4) For any $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$, $Z(\underbrace{g^a h^c, g^b h^d}_{\text{"}(g,h)\cdot\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\text{"}},\tau)$ is proportional to $Z(g,h,\underbrace{\tfrac{a\tau+b}{c\tau+d}}_{=M.\tau})$.

$SL_2(\mathbb{Z})$ mod. invariance

(5) $Z(g,h,\tau) = j(\tau) - 744 \iff g=h=1 \in \mathbb{M}$

(Obtain Monstrous Moonshine for $g=1$.)

- In 2003 Gerald Höhn resolved the Hauptmodul claim for the case where g is an element in the conjugacy class 2A in $\mathbb{M}$.

Moreover, he gave a general strategy for proving that Generalized Moonshine trace functions are Hauptmoduln.

- In 2015, Scott Carnahan finally proved the Gen. Moonshine Conj. following Höhn's approach.

One important ingredient is a result by van Ekeren, M., Scheithauer (see below)

<u>Idea of the proof:</u> (mainly the Hauptmodul claim)

- Consider the Monster VOA $V^\natural$ (of $c=24$)

  ↳ it is <u>holomorphic</u>, i.e. has only one irr. mod. up to iso, namely $V^\natural$ itself

  ↳ [DLM00] : $V^\natural$ has only one g-twisted module for each $g \in \mathrm{Aut}(V^\natural) \cong \mathbb{M}$

  Dong, Li, Mason

  graded in $\rho(V^\natural(g)) + \frac{\mathbb{Z}_{\geq 0}}{n}$

  $\leadsto$ call this module $V^\natural(g)$

  say: via ϕ

  and $C_{\mathbb{M}}(g)$ acts projectively on $V^\natural(g)$

> This is true for any „nice" holomorphic VOA V and not just for $V^\natural$. Indeed, (1), (2), (4), (5) are general claims about „nice" hol. VOAs and don't have much to do with the Monster.
> Claim (3) has a more exceptional, moonshine-like quality

- Consider the trace functions

$$T(v,g,h,\tau) := q^{-1} \sum_{k=0}^{\infty} \mathrm{tr}_{(V^\natural(g))_{\rho+\frac{k}{n}}} o(v)\,\phi(h)\, q^{\rho+\frac{k}{n}}$$

$$= q^{-\frac{c}{24}} \operatorname{tr}_{V^\natural(g)} O(v)\, \phi(h)\, q^{L_0}$$

$\frac{c}{24} = 1$ (sme $c=24$)

grading operator

(twisted one-point correlation functions)

- Special case: $v = |0\rangle \rightsquigarrow O(v) = \mathrm{id}$

$$Z(g,h,\tau) := T(|0\rangle, g, h, \tau) = q^{-\frac{c}{24}} \operatorname{tr}_{V^\natural(g)} \phi(h)\, q^{L_0}$$

- Claims (1), (2), (4), (5) were shown in [DLM00].
(But for (4) one needs a result from [Carnahan-Miyamoto-16])

<u>Def. of rule</u>: choose $V(g)$ as $V^\natural(g)$ and $Z(g,h,\tau)$ as $T(|0\rangle, g, h, \tau)$

<u>It remains to prove the Hauptmodul claim:</u>

- <u>Case 1</u>: If $g^a h^c$ is Fricke for some a, c with $(a,c)=1$, then $Z(g^a h^c, g^b h^d, \tau)$ is a Hauptmodul [! Claim !] for all b, d s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Hence $Z(g,h,\tau)$ is prop. to a mod. transform of $Z(g^a h^c, g^b h^d, \tau)$ and hence a Hauptmodul
- <u>Case 2</u>: If $g^a h^c$ is non-Fricke for all $(a,c)=1$, then it is easy to see that the tw. modules $V^\natural(g^a h^c)$ has L_0-spectrum ≥ 1 and hence $Z(g^a h^c, g^b h^d, \tau)$ is regular at cusp ∞ for all

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and here by mod. inv. $Z(g,h,\tau)$ is regular at all cusps

$\Rightarrow$ hence $Z(g,h,\tau)$ is constant.

Recall: Def. of Fricke. An element $g \in M$ is called Fricke if its McKay-Thompson series $T_g(\tau)$ is proportional to $T_g(-\frac{1}{N\tau})$ for some N. (Only depends on conjugacy class)

It remains to show: g Fricke element in the Monster, $h \in C_M(g)$.
Then $Z(g,h,\tau)$ is a Hauptmodul

$\Longrightarrow$ This is proved by Carnahan

Idea of proof:

- Monster VOA $V^\natural$, $g \in \mathrm{Aut}(V^\natural) \cong M$ Fricke element

$n = \mathrm{ord}(g)$.

fp. subVOA $(V^\natural)^g$ has n^2 irred. modules with group-like fusion and some fusion group F_n $\leftarrow$ some central extension $\mathbb{Z}_n . \mathbb{Z}_n$

$$\rightsquigarrow (V^\natural)^g(\alpha) \boxtimes (V^\natural)^g(\beta) \cong (V^\natural)^g(\alpha+\beta) \quad \text{for } \alpha, \beta \in F_n$$

The direct sum $\bigoplus_{\gamma \in F_n} (V^\natural)^g(\gamma)$ is an AIA with f.g.s. F_n — sum of all n^2 irred. untwisted $(V^\natural)^g$-modules

$$\| $$

$\bigoplus_{i=0}^{n} V^\natural(g^i)$ — sum of all n irred. g^i-tw. $V^\natural$-modules

$\hookrightarrow$ All this are results in [van Ekeren, M., Scheithauer]

- Consider a lattice L of genus $\mathrm{II}_{1,1}(F_u)$, i.e. $F_u \cong L'/L$ and the corresponding lattice AIA
$$\bigoplus_{\gamma+L\in L'/L} V_{\gamma+L}$$

- Define the graded tensor product
$$\mathcal{M} := \bigoplus_{\gamma+L\in L'/L} (V^{\natural})^{g}(z(\gamma+L)) \otimes V_{\gamma+L}$$
(„diagonal sum"), which is a weak VOA of central charge 26

- Quantisation: Apply BRST quantisation functor
$$\mathcal{M} \xmapsto{\text{BRST}} \mathfrak{m}_g = H^1_{\text{BRST}}(\mathcal{M})$$
rank 2, infinite-dim. Lie alg., equipped with a canonical proj. action of $C_u(g)$ by automorphisms

- Generate a Lie algebra L_g (a BKMA) whose denominator identity is some automorphic product. This Lie algebra has a nice, well-understood structure
- Show that $\mathfrak{m}_g \cong L_g \implies$ get Lie alg. with group action + nice shape
- Hauptmoduln conclusion: Use the twisted denominator identity to produce recursion relations on the characters that are strong enough to conclude that the characters are Hauptmoduln.