Vector valued modular forms and Eisenstein series

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Overview

1. Vector valued modular forms
2. Eisenstein series
3. Connections and applications

Discriminant group

- A lattice L is a free $\mathbb{Z}$-module of finite rank with a symmetric non-degenerate bilinear form $b : L \times L \to \mathbb{Q}$. We denote its type by (b^+, b^-).
- We call L *integral* if $b(L, L) \subseteq \mathbb{Z}$.
- We call L *even* if $b(x, x) \in 2\mathbb{Z}$ for all $x \in L$.
- The dual lattice L' is defined by

$$L' := \{x \in L \otimes_{\mathbb{Z}} \mathbb{R} \mid b(x, L) \subseteq \mathbb{Z}\}.$$

- We have $L \subseteq L'$ for integral L.
- For even lattices L'/L is called the *discriminant group* of L and b induces a non-degenerate form on L'/L taking values in $\mathbb{Q}/\mathbb{Z}$ as q, the associated quadratic form, does.
- $|L'/L| = |\det(L)|$, where $\det(L)$ is the determinant of the Gram matrix of L. In particular L'/L is finite.

Group algebra and Weil representation

- The group algebra of an even lattice L is $V := \mathbb{C}[L'/L]$, a $\mathbb{C}$-vectorspace with basis $\{\mathfrak{e}_\gamma \mid \gamma \in L'/L\}$.
- We endow it with the standard scalar product with respect to the basis $\{\mathfrak{e}_\gamma \mid \gamma \in L'/L\}$ (i.e. $\langle \mathfrak{e}_\gamma, \mathfrak{e}_\beta \rangle = \delta_{\gamma,\beta}$).
- Let $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle$.
- The assignment

$$\rho_L(T)\mathfrak{e}_\gamma := \mathrm{e}(q(\gamma))\mathfrak{e}_\gamma,$$

$$\rho_L(S)\mathfrak{e}_\gamma := \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|L'/L|}} \sum_{\delta \in L'/L} \mathrm{e}(-b(\gamma,\delta))\mathfrak{e}_\delta$$

extends for even type of L to the Weil representation $\rho_L : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(V)$, a unitary representation of $\mathrm{SL}_2(\mathbb{Z})$.

Petersson slash

- We denote by $M\tau$ the well known Möbius transformation on $\mathbb{H}$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ which is defined by $M\tau := \frac{a\tau+b}{c\tau+d}$.
- For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and a function f:$\mathbb{H} \to \mathbb{C}[L'/L]$ the Petersson slash $f \mid_\kappa M$ of weight $\kappa \in \mathbb{Z}$ is defined by

$$(f \mid_\kappa M)(\tau) := (c\tau + d)^{-\kappa} \rho_L(M)^{-1} f(M\tau).$$

- It defines a group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]^{\mathbb{H}}$.
- For unimodular lattices (i.e. $L' = L$) this simplifies to the well known scalar case:

$$(f \mid_\kappa M)(\tau) := (c\tau + d)^{-\kappa} f(M\tau).$$

Vector valued modular forms and cusp forms

Let $f \in \mathbb{C}[L'/L]^{\mathbb{H}}$ be $|_\kappa$ T invariant. Then for $\gamma \in L'/L$ the functions $g_\gamma(\tau) := \mathrm{e}(-q(\gamma)\tau) f_\gamma(\tau)$ are 1-periodic. So for holomorphic f there exists a Fourier representation

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+q(\gamma)} c(\gamma, n) \mathfrak{e}_\gamma(n\tau)$$

with

$$c(\gamma, n) := \int_0^1 f_\gamma(\tau)\, \mathrm{e}(-n\tau) dx.$$

Definition (Modular form and csup form)

A *modular form* (resp. *cusp form*) of weight $\kappa \in \mathbb{Z}$ is a holomorphic map $f : \mathbb{H} \to \mathbb{C}[L'/L]$ which is $\mathrm{SL}_2(\mathbb{Z})$ invariant with respect to $|_\kappa$ and satisfies $c(\gamma, n) = 0$ for all $n < 0$ (resp. $n \leq 0$).

Definition of Eisenstein series

- For $\beta \in L'/L$ with $\mathfrak{e}_\beta$ being $\Gamma_\infty := \langle \pm T \rangle$ invariant

$$E_\beta(\tau) := \frac{1}{2} \sum_{M \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathfrak{e}_\beta \mid_\kappa M$$

  defines a modular form of weight κ for $\kappa > 2$.
- But I was interested in Eisenstein series of weight $\kappa = 3/2$.
- Use Hecke's trick ($\tau = x + iy$):

$$E_\beta(\tau, s) := \frac{1}{2} \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} (\mathfrak{e}_\beta y^s) \mid_\kappa^* (M, \phi).$$

- $E_\beta(\tau, s)$ converges for $\Re(s) > 1 - \kappa/2$.
- Goal: Define $E_\beta(\tau, 0)$ for $\kappa = 3/2$.
- Path towards our goal: Compute Fourier expansion for $E_\beta(\tau, s)$ and compute the meromorphic continuation of the coefficients in s.

Fourier coefficients of $E_\beta(\tau, 0)$

Theorem (Bruinier and Kühn, 2003)

$$E_\beta(\tau, s) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} - q(\gamma)} c_\beta(\gamma, n, s, y) \mathfrak{e}_\gamma(nx)$$

with $c_\beta(\gamma, n, s, y)$ *given by*

$$\begin{cases} (\delta_{\beta,\gamma} + \delta_{-\beta,\gamma}) y^s + 2\pi y^{1-\kappa-s} \dfrac{\Gamma(\kappa + 2s - 1)}{\Gamma(\kappa + s)\Gamma(s)} \displaystyle\sum_{c \in \mathbb{Z} \setminus \{0\}} |2c|^{1-\kappa-2s} H_c^*(\beta, 0, \\ \dfrac{2^\kappa \pi^{s+\kappa} |n|^{s+\kappa-1}}{\Gamma(s+\kappa)} \mathcal{W}_s(4\pi n y) \displaystyle\sum_{c \in \mathbb{Z} \setminus \{0\}} |c|^{1-\kappa-2s} H_c^*(\beta, 0, \gamma, n), & n > 0, \\ \dfrac{2^\kappa \pi^{s+\kappa} |n|^{s+\kappa-1}}{\Gamma(s)} \mathcal{W}_s(4\pi n y) \displaystyle\sum_{c \in \mathbb{Z} \setminus \{0\}} |c|^{1-\kappa-2s} H_c^*(\beta, 0, \gamma, n), & n < 0. \end{cases}$$

Fourier coefficients of $E_0(\tau, 0)$ for $L = \mathbb{Z}$

Theorem

Let $L = \mathbb{Z}$ with $q(x) = Nx^2$ for $N \in \mathbb{N}$. The coefficients $C(\gamma, n, s)$ have for $\kappa > 0$ with $2\kappa \equiv 3 \pmod 4$ a meromorphic continuation on $\mathbb{C}$ in s which is holomorphic at $s = 0$. We have $C(\gamma, n, 0)$ equals

$$\begin{cases} \frac{(-1)^{(2\kappa+1)/4} 2^{\kappa+1/2} \pi^\kappa n^{\kappa-1}}{\Gamma(\kappa)\sqrt{N}} \frac{L(\chi_{D_0}, \kappa - 1/2)}{\zeta(2\kappa - 1)} \sigma_{\gamma,n}(0), & \text{if } n > 0, \\ \frac{(-1)^{(2\kappa+1)/4} 2^{\kappa-1/2} \pi^\kappa |n|^{\kappa-1}}{\sqrt{N}\zeta(2\kappa - 1)} \sigma_{\gamma,n}(0), & \text{if } n < 0, \kappa = \frac{3}{2} \text{ ar} \\ 0, & \text{else.} \end{cases}$$

$$\sigma_{\gamma,n}(s) := \prod_p L^{(p)}_{\gamma,n}(p^{1/2-\kappa-2s}) \frac{1 - \chi_{D_0}(p) p^{1/2-\kappa-2s}}{1 - p^{1-2\kappa-4s}}.$$

My result (for the special case N square-free)

Theorem (Buck, 2016)

Let $N \in \mathbb{N}$ square-free, L as above and $\kappa = 3/2$. Then we have:

$$E_0(\tau, 0) = \left(2 - \frac{6 \cdot 2^{\#(N)} \sqrt{N}}{\pi \sqrt{y}\, \sigma_1(N)}\right) \mathfrak{e}_0$$
$$- \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} - q(\gamma) \\ n > 0}} \frac{24}{\sigma_1(N)} \sum_{d | (z, r, N)} dH\left(\frac{4Nn}{d^2}\right) \mathfrak{e}_\gamma(n\tau)$$
$$- \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} - q(\gamma) \\ -4Nn = \square \\ n < 0}} \frac{6 \cdot 2^{\#(N,r)} \sqrt{-4Nn}}{\sigma_1(N)\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}, -4\pi n y\right) \mathfrak{e}_\gamma(n\tau).$$

Zagier's Eisenstein series

Theorem

Let $L = \mathbb{Z}$ with $q(x) := Nx^2$, $N \in \mathbb{N}$ and $f \in H_{\kappa,\rho_L^*}$ be a weak Maass form of half integral weight $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Then

$$g(\tau) := \sum_{\gamma \in L'/L} f_\gamma(4N\tau)$$

is a scalar valued weak Maass form for $\Gamma_0(4N)$ of weight κ.

Using this translation $E_0(\tau, 0)$ for $N = 1$ and $\kappa = 3/2$ is mapped to a multiple of Zagier's Eisenstein series which he discovered in 1975:

$$\mathcal{F}(\tau) := \sum_{n=0}^{\infty} H(n) e^{2\pi i n\tau} + y^{-1/2} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 y) e^{-2\pi i n^2 \tau}.$$

with $\beta(y) := \frac{1}{16\pi} \int_1^\infty t^{-1/2} e^{-yt} \frac{dt}{t}$.

Further connections and applications

Jacobi Eisenstein series

- There is an isomorphism between vector valued modular forms with lattice $L = \mathbb{Z}$ and Jacobi forms.
- It turned out that our Eisenstein series correspond to Jacobi Eisenstein series.

Symmetrized Jacobi theta series

- Bruinier and Funke defined $\xi_\kappa : H_{\kappa,\rho_L^*} \longrightarrow M^!_{2-\kappa,\rho_L}$ which is antilinear and surjective and has kernel $M^!_{\kappa,\rho_L^*}$.
- $\xi_\kappa(E_0(\tau,0))$ for $\kappa = 3/2$ and N square-free is the symmetrized Jacobi theta series of weight $1/2$.
- This gives rise to a formular involving Hurwitz class numbers of the regularised Petersson scalar product of weakly holomorphic modular forms of weight $1/2$ with the symmetrized Jacobi theta series.

Thank you for your attention!