Realizing elliptic and hyperbolic Eisenstein series as theta lifts

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Introduction

Recall the following non-holomorphic Eisenstein series:

$$E_p(z,s) = \sum_{M \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\sigma_p^{-1} M z)^s,$$

$$E_c^{\mathrm{hyp}}(z,s) = \sum_{M \in \Gamma_c \backslash \mathrm{SL}_2(\mathbb{Z})} \cosh\left(d_{\mathrm{hyp}}(Mz, c)\right)^{-s},$$

$$E_w^{\mathrm{ell}}(z,s) = \sum_{M \in \Gamma_\omega \backslash \mathrm{SL}_2(\mathbb{Z})} \sinh\left(d_{\mathrm{hyp}}(Mz, w)\right)^{-s},$$

for a cusp p, a geodesic c, and a point $w \in \mathbb{H}$, respectively.

Goal

Realize the elliptic and hyperbolic Eisenstein series as theta lifts to (re)prove known and new properties of them.

The general setting

We want to use the vector valued theta lift:

- Let (L, q) be an even lattice of signature (b^+, b^-).
- Let L' be the dual lattice of L. Then L'/L is the discriminant group of L (a finite abelian group).
- The group algebra $\mathbb{C}[L'/L]$ is a finite-dimensional $\mathbb{C}$-vector space with formal bases elements $\mathfrak{e}_\gamma$, $\gamma \in L'/L$.
- The Weil-representation ρ_L associated to L is a unitary representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$.
- Given a real analytic function $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ we define

$$F|_{k,L}M = (cz + d)^{-k}\rho_L(M)^{-1}F(Mz), \quad M \in \mathrm{SL}_2(\mathbb{Z}).$$

  We call F modular of weight k (with respect to ρ_L) if $F = F|_{k,L}M$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$.

The regularized theta lift

Definition

Let F be modular of weight $k = \frac{b^+ - b^-}{2}$ w.r.t ρ_L. The theta lift of F is given by

$$\Phi_L[F](v) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} \langle F(z), \Theta_L(z,v)\rangle \, \mathrm{Im}(z)^{b^+/2} \, d\nu(z).$$

Here $\Theta_L(z,v)$ is Siegel's vector valued Theta function:

$$\Theta_L(z,v) = \sum_{\gamma \in L'/L} \sum_{\lambda \in \gamma + L} e(xq(\lambda) + iy(q(\lambda_v) - q(\lambda_{v^\perp})))\mathfrak{e}_\gamma$$

for $z = x + iy \in \mathbb{H}$, $v \in \mathrm{Gr}(L) =$ space of b^+-dimensional positive definite subspaces of $L \otimes \mathbb{R}$.

The regularized theta lift

In what sense is

$$\Phi_L[F](v) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} \langle F(z), \Theta_L(z,v)\rangle \operatorname{Im}(z)^{b^+/2}\, d\nu(z)$$

modular in v?

- $\mathrm{Gr}(L) \circlearrowleft \mathrm{SO}_0(q)$ = component of the special orthogonal group of q, that contains the identity
- $\mathrm{SO}_0(q,\mathbb{Z})$ = subgroup of automorphisms of L
- $\Theta_L(z,v)$ is $\mathrm{SO}_0(q,\mathbb{Z})$-invariant in v, that is

$$\Theta_L(z,\alpha v) = \Theta_L(z,v) \qquad \text{for all } \alpha \in \mathrm{SO}_0(q,\mathbb{Z})$$

- $\Phi_L[F](v)$ is $\mathrm{SO}_0(q,\mathbb{Z})$-invariant

Lifting in signature $(2, 1)$

- Let $L = \{\lambda \in \mathbb{Z}^{2\times 2} \colon \operatorname{tr}(\lambda) = 0\}$, $q(\lambda) = -\det(\lambda)$.
- (L, q) has signature $(2, 1)$ and

$$\operatorname{Gr}(L) \simeq \mathbb{H}, \qquad \operatorname{SO}_0(q) \simeq \operatorname{SL}_2(\mathbb{R}), \qquad \operatorname{SO}_0(q, \mathbb{Z}) \simeq \operatorname{SL}_2(\mathbb{Z}).$$

- Thus given F of weight $1/2$ the lift $\Phi_L[F](v)$ is a classical non-holomorphic modular form of weight 0.

Goal (revisited)

Find an input F of weight $1/2$ such that $\Phi_L[F](v)$ is an elliptic or hyperbolic Eisenstein series!

Recall that this is known for the classical (parabolic) Eisenstein series

$$E_\infty(z, s) = \sum_{M \in \Gamma_\infty \backslash \operatorname{SL}_2(\mathbb{Z})} \operatorname{Im}(Mz)^s.$$

Lifting in signature $(2, 1)$

Define the vector valued non-holomorphic Eisenstein series of weight $1/2$ as

$$E_\infty^L(z,s) = \sum_{M \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(z)^s \mathfrak{e}_0 \Big|_{1/2,L} M.$$

Theorem

We have

$$\Phi_L[E_\infty^L(\cdot, s)](\tau) \doteq E_\infty(\tau, 2s).$$

Idea of proof:

- Unfolding yields

$$\Phi_L[E_\infty^L(\cdot, s)](\tau) \doteq \sum_{\substack{\lambda \in L \\ q(\lambda)=0}} q(\lambda_{v(\tau)})^{-s}.$$

- Note that $q(\lambda_{v(\tau)}) = \frac{1}{4}\,\mathrm{Im}(\gamma_\lambda \tau)^{-2}$.

Lifting in signature $(2,1)$

More generally, one can show that:

Lemma

Let $\lambda \in L$ with $q(\lambda) = m$ and $\tau \in \mathbb{H}$. Then

$$q(\lambda_{v(\tau)}) = \begin{cases} m \cosh^2(d_{\mathrm{hyp}}(\tau, c_\lambda)), & \text{if } m > 0, \\ \frac{1}{4}\operatorname{Im}(\sigma_\lambda^{-1}\tau)^{-2}, & \text{if } m = 0, \\ |m| \sinh^2(d_{\mathrm{hyp}}(\tau, \tau_\lambda)), & \text{if } m < 0, \end{cases}$$

where c_λ = geodesic, σ_λ = scaling matrix, τ_λ = CM point, each associated to λ.

Idea: Use a modified Eisenstein series with an extra weight m in order to obtain elliptic and hyperbolic Eisenstein series.

Realizing averaged Eisenstein series

Define the weighted Poincaré series

$$U_m(z,s) = \sum_{M\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(z)^s e(mz)\mathfrak{e}_0 \Big|_{1/2,L} M.$$

Theorem

We have

$$\Phi_L[U_m(\cdot,s)](\tau) \doteq \begin{cases} \sum_{\lambda\in\mathrm{SL}_2(\mathbb{Z})\backslash L_m} E^{\mathrm{hyp}}_{c_\lambda}(\tau,2s), & \text{for } m>0, \\ \sum_{l\in\mathrm{SL}_2(\mathbb{Z})\backslash P_1(\mathbb{Q})} E_l(\tau,2s), & \text{for } m=0, \\ \sum_{\lambda\in\mathrm{SL}_2(\mathbb{Z})\backslash L_m} E^{\mathrm{ell}}_{\tau_\lambda}(\tau,2s), & \text{for } m<0. \end{cases}$$

Here the sums are all finite and run over all geodesics c_λ of norm m, all cusps p, and all CM points τ_λ of norm m, respectively.

Realizing an arbitrary elliptic Eisenstein series

Question

Is it also possible to obtain a **single** *hyperbolic or elliptic Eisenstein series as a theta lift?*

Define another weighted Poincaré series by

$$F_m^L(z,s) = \sum_{M \in \Gamma_\infty \setminus \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(z)^s \mathcal{M}_{k,s}(4\pi|m|y) e(mx) \mathfrak{e}_0 \Big|_{k,L} M,$$

where $z = x + iy$, $\mathcal{M}_{k,s}(y) =$ modified M-Whittaker function.

Theorem

Let L be a certain 'nice' lattice of signature $(2,2)$. Then

$$\Phi_L[F_{-1}^L(\cdot,s)](\tau,\omega) \doteq E_\omega^{\mathrm{ell}}(\tau,2s)$$

for any $\omega \in \mathbb{H}$.

Motivation for the hyperbolic case

Question

Is there a connection between the two lifting results for the elliptic Eisenstein series?

Let L be a lattice and $m < 0$. Then

$$F_m^L(z,s,t) = \sum_{M \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(z)^s \mathcal{M}_{k,t}(4\pi|m|y) e(mx) \mathfrak{e}_0 \Big|_{k,L} M.$$

So for $m < 0$, $t = k/2$ and L being the lattice of signature $(2,1)$ from before we get back the series $U_m(z,s)$!

Similarly, the Poincaré series

$$G_m^L(z,s,t) = \sum_{M \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(z)^s \mathcal{W}_{k,t}(4\pi|m|y) e(mx) \mathfrak{e}_0 \Big|_{k,L} M,$$

specializes to $U_m(z,s)$ for $m > 0$, $t = 1 - k/2$ and L as before.

Realizing an arbitrary hyperbolic Eisenstein series?

Question

Can we find a lattice L of some signature such that the lift $\Phi_L[G_m^L(\cdot, s, t)](v)$ of

$$G_m^L(z, s, t) = \sum_{M \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(z)^s \mathcal{W}_{k,t}(4\pi |m| y) e(mx) \mathfrak{e}_0 \Big|_{k,L} M,$$

realizes an arbitrary hyperbolic Eisenstein series?

Remarks:

- $\mathrm{Gr}(L)$ should look like (or contain) the space

$$\mathbb{H} \times \{\text{geodesics in } \mathbb{H}\}$$

- Is there a natural choice for the parameter t?