BRST Construction of 10 Borcherds-Kac-Moody Algebras

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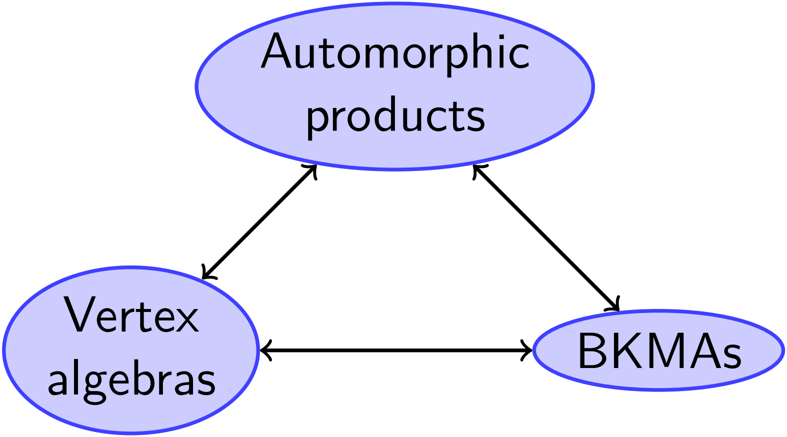
Introduction



Contents

Section 1

Borcherds-Kac-Moody Algebras

Kac-Moody Algebras

- Natural generalisations of finite-dimensional simple Lie algebras, ususally infinite-dimensional
- Defined by generators and relations through a generalized Cartan matrix (not positive definite)
- They have: Weyl group, Weyl character formula, Cartan subalgebra, roots, weights, etc.
- Examples:
  - finite-dimensional simple Lie algebras,
  - (twisted) affine Lie algebras [Kac90]

Borcherds-Kac-Moody Algebras

- Further weaken the conditions on the Cartan matrix: allow imaginary simple roots
- They still have: Weyl group, Weyl character formula, Cartan subalgebra, roots, weights, etc.
- Examples:
  - Fake Monster Lie algebra [Bor90],
  - Monster Lie algebra [Bor92],
  - Fake Baby Moster Lie algebra [HS03],
  - Baby Monster Lie algebra [Hö03],
  - 10 BKMAs from [Sch04, Sch06]

Modular Forms I

- g element of squarefree order m in $M_{23} \subset \mathrm{Co}_0 = \mathrm{Aut}(\Lambda)$
- 10 conjugacy classes with $m = 1, 2, 3, 5, 6, 7, 11, 14, 15, 23$
- g has cycle shape $\prod_{t|m} t^{24/\sigma_1(m)}$
- Consider the eta product

$$\eta_g(\tau) = \prod_{t|m} \eta(t\tau)^{24/\sigma_1(m)}$$

- Cusp form for $\Gamma_0(m)$ of weight $w := 12\sigma_0(m)/\sigma_1(m)$

Modular Forms II

- Lift $f_g(\tau) = 1/\eta_g(\tau)$ to vector-valued modular form [Sch06]

$$F_g(\tau) = \sum_{M \in \Gamma_0(m) \backslash \Gamma} f_g|_M(\tau) \bar{\rho}_D(M^{-1}) \mathfrak{e}^0$$

  of weight $-w$ for the dual Weil representation $\bar{\rho}_L$ of lattice $L = \Lambda^g \oplus II_{1,1}(m)$
- Apply Borcherds lift [Bor92] to obtain completely reflective automorphic product Ψ_g of singular weight

Summary

$$g \mapsto 1/\eta_g \mapsto F_g \mapsto \Psi_g$$

Borcherds-Kac-Moody Algebras

- Expansion of Ψ_g at any cusp

$$\mathfrak{e}^{\rho} \prod_{d|m} \prod_{\alpha \in (L' \cap L/d)^+} (1 - \mathfrak{e}^{\alpha})^{[1/\eta_g](-d\langle\alpha,\alpha\rangle/2)} = \sum_{w \in W} \det(w) w(\eta_g(\mathfrak{e}^{\rho}))$$

- Denominator identities of 10 BKMAs of rank $k = 2w + 2$ whose roots lie in L'
- Classification result [Sch06] ([Mö12]): these are all BKMAs whose denominator identities are completely reflective automorphic products of singular weight on lattices of squarefree level

Goal

Realise these 10 BKMAs uniformly as physical states of bosonic strings moving on suitable spacetimes.

Section 2

BRST Quantisation

Lie Algebra

BRST Quantisation

Certain VA M of $c = 26$ $\overset{BRST}{\rightsquigarrow}$ BKMA $\mathfrak{g} = H^1_{\text{BRST}}(M)$.

- BRST operator $Q = j_0^{\text{BRST}}$ on $W = M \otimes V_{\text{gh}}$.

$$Q^2 = 0, \quad [Q, L_0] = 0, \quad [Q, U] = U$$

- BRST cochain complex

$$\ldots \xrightarrow{Q} W_n^{u-1} \xrightarrow{Q} W_n^u \xrightarrow{Q} W_n^{u+1} \xrightarrow{Q} \ldots$$

- cohomological spaces (exact for $n \neq 0$)

$$H^\bullet = H_0^\bullet$$

- $\mathfrak{g} := H^1 = H^1_{\text{BRST}}(M)$ is a Lie algebra under $[u, v] = (b_0 u)_0 v$ [LZ93]

Vanishing Theorem I

- Suppose $V_L \subseteq M$ for even Lorentzian lattice of rank $k \geq 2$
- Assume $U := \mathrm{Com}_M(V_L)$ and V_L form a Howe pair in M
- Suppose $U_1 = 0$
- $U \otimes V_L$-module decomposition

$$M \cong \bigoplus_{\alpha+L \in L'/L} U(\alpha + L) \otimes V_{\alpha+L}$$

- $M = \bigoplus_{\alpha \in L'} M(\alpha)$ is naturally graded by L'

Theorem (Vanishing Theorem [Fei84, FGZ86])

Let $\alpha \neq 0$. Then $H^1(\alpha) = H^2(\alpha)$ and $H^u(\alpha) = 0$ for $u \neq 1, 2$.

Vanishing Theorem II

- Euler-Poincaré characteristic for $\alpha \neq 0$:

$$\dim(H^1(\alpha)) = \left[\mathrm{ch}_{U(\alpha+L)}(q)/\eta(q)^{k-2}\right](-\langle\alpha,\alpha\rangle/2)$$

- Direct computation: $H^1(0) \cong L \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}$
- $\mathfrak{g} = H^1$ has self-centralising subalgebra $\mathfrak{g}(0)$
- CSA $\mathfrak{g}(0) \cong \mathfrak{h}$ acts as $\langle\cdot,\alpha\rangle$ on $\mathfrak{g}(\alpha)$
- Finite-dimensional root spaces $\mathfrak{g}(\alpha)$, $\alpha \in L' \setminus \{0\}$
- L'-grading: $[\mathfrak{g}(\alpha), \mathfrak{g}(\beta)] \subseteq \mathfrak{g}(\alpha+\beta)$
- Sometimes: $\mathfrak{g}$ is a BKMA

Section 3

Application

Construction of 10 BKMAs I

- Consider for $K = II_{1,1}(m)$ the vertex algebra

$$M_g = \bigoplus_{\gamma+K \in K'/K} V_\Lambda^{\hat{g}}(\gamma + K) \otimes V_{\gamma+K}$$

(use orbifold theory [EMS15])

- Use $V_N^{\hat{g}} \otimes V_{\Lambda^g} \subseteq V_\Lambda^{\hat{g}}$ where $N = (\Lambda^g)^\perp$:

$$M_g \cong \bigoplus_{\alpha+L \in L'/L} V_N^{\hat{g}}(\alpha + L) \otimes V_{\alpha+L}$$

with $L = \Lambda^g \oplus II_{1,1}(m)$

- BRST quantisation: $\mathfrak{g} = H^1_{\text{BRST}}(M_g)$

Construction of 10 BKMAs II

Theorem (M.)

$$\mathrm{ch}_{V_N^{\hat{g}}(\alpha+L)}(\tau)/\eta(\tau)^{\mathrm{rk}(\Lambda^g)} = (F_g)_{\alpha+L}(\tau).$$

Corollary

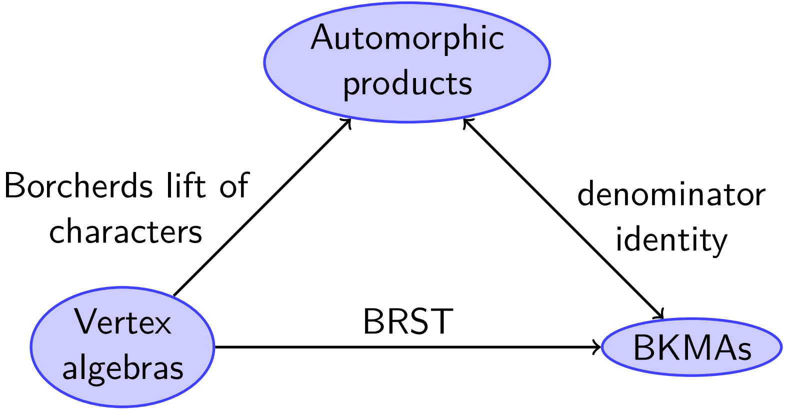
For $\alpha \neq 0$:

$$\dim(\mathfrak{g}(\alpha)) = [(F_g)_{\alpha+L}](-\langle\alpha,\alpha\rangle/2) = \sum_{d|m} \delta_{\alpha\in L'\cap\frac{1}{d}L}\left[\frac{1}{\eta_g}\right]\left(-d\frac{\langle\alpha,\alpha\rangle}{2}\right).$$

Conjecture

In each of the 10 cases $\mathfrak{g}$ is a BKMA. Get exactly those from [Sch06].

Summary


Automorphic products
Borcherds lift of characters
denominator identity
Vertex algebras
BRST
BKMAs

Thank you for your attention!

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