Hecke operators on vector valued modular forms

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Abstract Hecke theory

- Let G be a group and $\Delta \subseteq G$ a subsemigroup and $\Gamma \subseteq \Delta \subseteq G$ a subgroup of finite index. Denote by $\mathcal{R}(\Gamma, \Delta)$ the free $\mathbb{Z}$-module over $\{\Gamma\alpha\Gamma : \alpha \in \Delta\}$ with canonical multiplication.

  Assume Δ has a right action $\sigma : \mathcal{R}(\Gamma, \Delta) \times M \to M$, $(\delta, m) \mapsto m^{\delta}$ on a $\mathbb{Z}$-module M, i.e.
  - the map $m \mapsto m^{\delta}$ is a $\mathbb{Z}$-endomorphism for every $\delta \in \Delta$
  - $(m^{\gamma})^{\delta} = m^{\gamma\delta}$ for all $\gamma, \delta \in \Delta$
  - $m^1 = m$ for the unity $1 \in \Delta$
- Define double coset operators on the fixed point set

$$M^{\Gamma} = \{m \in M : \ m^{\gamma} = m \text{ for all } \gamma \in \Gamma\} :$$

  Let $\alpha \in \Delta$ and decompose the double coset $\Gamma\alpha\Gamma = \bigcup_{i \in I} \Gamma\delta_i$. The double coset operator $T(\Gamma\alpha\Gamma)$ is defined as

$$\begin{array}{rccc} T(\Gamma\alpha\Gamma) : & M^{\Gamma} & \to & M^{\Gamma} \\ & m & \mapsto & \sum_{i \in I} m^{\delta_i}. \end{array}$$

Scalar valued Hecke theory

- For $\Gamma = SL_2(\mathbb{Z})$ and suitable choices of $G, M, \ldots$ and $\sigma(\alpha) = |_\alpha$, we have $M^\Gamma = \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$
- Double coset operators

$$T(\mathrm{SL}_2(\mathbb{Z})\alpha\,\mathrm{SL}_2(\mathbb{Z}))F \doteq \sum_{\delta \in S_\alpha} F|_\delta$$

  map $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ and the subspace $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ of cusp forms to itself.
- Hecke operators:

$$T(m) = \sum_{\delta \in S_m} T(\mathrm{SL}_2(\mathbb{Z})\delta\,\mathrm{SL}_2(\mathbb{Z}))$$

Hecke theory for vector valued modular forms

Let D be a discriminant form of odd level N and even signature with Weil representation ρ_D and $\mathcal{M}_k(\mathbb{C}[D])$ the space of vector valued modular forms of weight $k \in \mathbb{Z}$.

- Interpret $\mathcal{M}_k(\mathbb{C}[D]) = M^{\mathrm{SL}_2(\mathbb{Z})}$ as fixed point set of a suitable module M under the action of $\sigma(\alpha)F = \rho_D(\alpha)^{-1}F|_\alpha$.
- Construction of double coset operators on vector valued modular forms by abstract Hecke theory:
  Find Hecke algebra $\mathcal{R}(\mathrm{SL}_2(\mathbb{Z}), \Delta)$ with $\mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \Delta$ and right action

$$\sigma : \mathcal{R}(\mathrm{SL}_2(\mathbb{Z}), \Gamma) \times \mathcal{M}_k(\mathbb{C}[D]) \to \mathcal{M}_k(\mathbb{C}[D])$$

  with $\sigma(\alpha)(F) = \rho_D(\alpha)^{-1}F|_\alpha$ for $\alpha \in \mathrm{SL}_2(\mathbb{Z})$

Possible approaches

- Natural idea: Choose $\Delta := \{\alpha \in \mathrm{Mat}^{2\times 2}(\mathbb{Z}) : \det(\alpha) > 0\}$; consider ρ_D as a representation of the finite group

$$\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \cong \mathrm{SL}_2(\mathbb{Z}_N);$$

  for $(m, N) = 1$, extend ρ_D to $\mathrm{GL}_2(\mathbb{Z}_N)$ and use

$$\mathbb{T}_m := \{\alpha \in \mathrm{Mat}^{2\times 2}(\mathbb{Z}) : \det(\alpha) = m\} \xrightarrow{\pi} \mathrm{GL}_2(\mathbb{Z}_N)$$

  and define $\sigma(\alpha) = \rho_D(\pi^{-1}(\alpha))^{-1}(\cdot)|_\alpha \to$ Not possible in general
- Choose different Hecke algebra!

Approach by Bruinier and Stein

Bruinier and Stein:

$$\Delta = \{(\alpha, x) \in \mathrm{Mat}^{2\times 2}(\mathbb{Z}) \times \mathbb{Z}_N^* : \det(\alpha) \equiv x^2 \pmod{N}\}$$

and define

$$\sigma(\alpha, x)F := \rho_D\left(\pi^{-1}(x^{-1}\alpha)\right) F|_\alpha$$

- Corresponding double coset operators $T(\alpha) = T(\mathrm{SL}_2(\mathbb{Z})\alpha\,\mathrm{SL}_2(\mathbb{Z}))$ map cusp forms to cusp forms and are self-adjoint; they satisfy

  $$T(\alpha) \circ T(\beta) = T(\alpha\beta) = T(\beta) \circ T(\alpha)$$

  for $(\det(\alpha), \det(\beta)) = 1$.

- Double coset operators for $\left(\frac{\det(\alpha)}{N}\right) = -1$ missing!

Generalization by Werner

Ideas:

- Translate determinant to a square mod N by multiplying with suitable matrix
- Allow right action to change the discriminant form to compensate for translation
- Enlarge underlying vector space to get meaningful right action

Generalization by Werner

- Let $t \in \mathbb{Z}_N^*$. By tD denote D with quadratic form $\gamma \mapsto t \cdot \gamma^2/2$ with the canonical map $\mathcal{M}^t : \mathbb{C}[D] \to \mathbb{C}[tD]$
- Choose

$$\Delta = \{(\alpha, t, x) \in \mathrm{Mat}^{2\times 2}(\mathbb{Z}) \times \mathbb{Z}_N^* \times \mathbb{Z}_N^* : \ t \det(\alpha) \equiv x^2 \pmod{N}\}$$

and let the corresponding Hecke algebra act on

$$X = \bigoplus_{t \in \mathbb{Z}_N^*} \mathcal{M}_k(\mathbb{C}[tD])$$

by the right action of

$$\sigma(\alpha, t, x)F := \mathcal{M}^t \left(\oplus_{t \in \mathbb{Z}_N^*} \rho_{tD}\right) \left(\pi^{-1}(x^{-1}\alpha\epsilon_t)\right) F|_\alpha$$

with $\epsilon_t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_N)$

Generalization by Werner

- Induced double coset operator $T^{(t,x)}(\alpha)$ on direct summand $\mathcal{M}_k(D)$:

$$\begin{aligned} T^{(t,x)}(\alpha): \quad \mathcal{M}_k(D) &\to \mathcal{M}_k(tD) \\ F &\mapsto \mathcal{M}^t\left(m^{k/2-1}\sum_{\delta\in S_\alpha}\rho_D\left(\pi^{-1}(x^{-1}\delta\epsilon_t)\right)^{-1}F|_\delta\right) \end{aligned}$$

  and the m-th Hecke operator is $T^{(t,x)}(m) := \sum_{\delta\in S_m} T^{(t,x)}(\delta)$.
- Hecke relations:
  For $(m,n)=1$, p prime and suitable $t,s,x,y\in\mathbb{Z}_N^*$, we have

$$T^{(ts,xy,1)}(mn) = T^{(t,x,s)}(m)\,T^{(s,y,1)}(n) = T^{(s,y,t)}(n)\,T^{(t,x,1)}(m)$$

  and

$$T^{(ts,xy,1)}\left(p^{e+1}\right) = T^{(s,y,t)}(p)\,T^{(t,x,1)}\left(p^{e}\right) - p^{k-1}\,T^{(t/s,x/y,s^2)}\left(p^{e-1}\right)\Phi_s.$$

Generalization by Werner

Let $F = \sum_{\gamma \in D} F_\gamma \mathbf{e}^\gamma \in \mathcal{M}_k(\mathbb{C}[D])$ with $F_\gamma \in \Gamma(N)$.

- Hecke operators almost coincide with the scalar Hecke operators for $\Gamma(N)$:
$$T^{(t,x)}(m)F = \chi_D(x) \sum_{\gamma \in tD} \left(T^{\Gamma(N)}(m) F_{x\gamma} \right) \mathbf{e}^\gamma$$
- Hecke operators and lift almost commute: For $f \in \mathcal{M}_k(\Gamma(N))$ we have
$$T^{(t,x,\omega)}(m)\mathcal{L}_{[1,\gamma]}(f) = \mathcal{L}_{[t,x\gamma]} T^{\Gamma(N)}(m)(f)$$
- Action on Fourier coefficients:
If
$$F_\gamma = \sum_{n=0}^{\infty} c_n(F_\gamma)\, q^{n/N},$$
we have
$$c_n\left((T^{(t,x)}(m))_\gamma \right) = \chi_D(x) \sum_{d|(m,n)} \chi_{tD}(d) d^{k-1} c_{\frac{nm}{d^2}} \left(F_{d^{-1}x\gamma} \right)$$

Applications: Siegel-Weil formula

Theorem

If weight $k > 3$ (+technical conditions), vector valued Eisenstein series $E_{\{0\}}$ and the symmetrized genus theta series $\Theta_{gen,sym}$ satisfy

$$E_{\{0\}} = \left(\sum_{M \in \mathrm{Gen}(L)/\sim} \frac{|\operatorname{Aut} D|}{|\operatorname{Aut} M|} \right)^{-1} \cdot \Theta^{L}_{gen,sym}$$

Idea of proof:

- $E_{\{0\}}$ and $\Theta_{gen,sym}$ are eigenfunctions of $T^{(1,x)}(p)$ to the same eigenvalue $\lambda_p = p^{k-1} - 1$
- the eigenvalues of nonzero cusp forms for $T^{(1,x)}(p)$ are bounded by $|\lambda_p| \leq p^{k/2}(1 - \frac{1}{p}) < p^{k-1} - 1$

Problem: Previous construction relies on $(m, N) = 1$
Idea: Forget about abstract Hecke theory, define double coset operators explicitly.

- Define right action for simple representative of double cosets and continue canonically to the double coset: By elementary divisor theorem, it is sufficient to do this for $\alpha = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ with $a, d \in \mathbb{N}$ and $d|a$
- If $(m, N) > 1$, expect Hecke operators to reduce the size of discriminant form. Allow scaling of the discriminant form by arbitrary $t \in \mathbb{Z}_N$.

Defining the right action

Let D a discriminant form of odd level $N \in \mathbb{N}$ and $t \in \mathbb{Z}_N$.
The set

$$D^t = \{\gamma \in D : \gamma = t\delta \text{ for a } \delta \in D\}$$

with quadratic form $\gamma = t\delta \mapsto t \cdot {}^{\delta^2}/_2$ is again a discriminant form.

- Canonical map: $\mathcal{M}_t : D \to D^t$, $\gamma \mapsto t\gamma$
- By analogy to previous construction of double coset operators: For $a, d \in \mathbb{N}$ with $d|a$, the right action of $\alpha = \left(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right), t, x\right)$ is $\mathcal{M}_t$

Continuing the right action to double cosets

Lemma

Let $a, d \in \mathbb{N}$ *with* $d|a$ *and* $t, x \in \mathbb{Z}_N$, *satisfying*

(i) $t \in \mathbb{Z}_N^*$ *with* $t^{a/d} = x^2 \mod N$ *or*

(ii) $t = x = {}^a/_d \in \mathbb{N}$ *and "technical condition"*

For $\delta = \beta\alpha\beta' \in \mathrm{SL}_2(\mathbb{Z}) \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mathrm{SL}_2(\mathbb{Z})$ *we define*

$$\sigma(\delta, t, x)F := \rho_{D^t}^{-1}(\beta) \circ \mathcal{M}_t \circ \rho_D(\beta')^{-1} F_\delta .$$

This is independent of the chosen decomposition.

Conditions are not overly restrictive:

- If ${}^a/_d = p = x$ prime and $p^2 \nmid N$, there is $t \in \mathbb{Z}_N^*$, satisfying (i)

Hecke theory in the general case

Theorem

Let D be a discriminant form of odd level $N \in \mathbb{N}$. Let $\alpha = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ with $d|a$ and let $k \in \mathbb{N}$ and $t, x \in \mathbb{Z}_N$ satisfying one of the previous conditions. Then

$$\begin{array}{rccc} T^{(t,x)}(\alpha): & \mathcal{M}_k(D) & \to & \mathcal{M}_k(D^t) \\ & F & \mapsto & det(\alpha)^{k/2-1} \sum_{\delta \in S_\alpha} \sigma(\delta, t, x) F, \end{array}$$

maps cusp forms to cusp forms and satisfies

$$T^{(t,x,s)}(\alpha) \circ T^{(s,y,1)}(\beta) = T^{(ts,xy,1)}(\alpha\beta) = T^{(s,y,t)}(\beta) \circ T^{(t,x,1)}(\alpha)$$

for $(\det(\alpha), \det(\beta)) = 1$ *and suitable* $s, y \in \mathbb{Z}_N$.

- If $t, x \in \mathbb{Z}_N^*$: Coincides up to a character with Hecke theory of Werner

Outlook

Problem:
No suitable choice of $t, x \in \mathbb{Z}_N$ for arbitrary discriminant form and $m \in \mathbb{N}$

Possible approach: Map modular forms of arbitrary level to p-squarefree level:

- Choose maximal isotropic subgroup $H \subseteq D$ in p-Jordan component; level of $D_H = H^\perp / H$ is p-squarefree.
- Use canonical map

$$\begin{array}{rccc} \downarrow_H\colon & \mathcal{M}_k(D) & \to & \mathcal{M}_k(D_H) \\ & F = \sum\limits_{\gamma \in D} F_\gamma e^\gamma & \mapsto & \sum\limits_{\mathfrak{a} \in D_H} \left(\sum\limits_{\gamma \in \mathfrak{a}} F_\gamma \right) e^{\mathfrak{a}} \end{array}$$

$\rightarrow$ Combining with previously constructed operators might yield complete set of well-behaved Hecke operators for arbitrary discriminant forms