Theta Functions from L^1 Function

§0 Intro.

Theta func. are examples of modular forms.

All you need are

- (L, Q) even lattice
- $\varphi_\tau : L \otimes \mathbb{R} \longrightarrow \mathbb{C}$ suitable

Def 0:

$$\Theta(\varphi_\tau, L) := \sum_{h \in L^*/L} e_h \sum_{\lambda \in L+h} \varphi_\tau(\lambda)$$

Key point: φ_τ satisfies

① $\varphi_{\tau+1}(x) = e(Q(x)) \cdot \varphi_\tau(x)$

② $\mathcal{F}(\varphi_{-1/\tau})(x) = \sqrt{-i\tau} \cdot \varphi_\tau(x)$

$$\mathcal{F}(f)(x) := \int_{\mathbb{R}} f(t)\, e(tx)\, dt$$

③ $\sum_{\lambda \in L+\varepsilon} \varphi_\tau(\lambda)$ converges uniformly for $\varepsilon \in L \otimes \mathbb{R}$.

Goal: consider an L^1 function $\tilde{\varphi}_\tau(x)$, and construct $\Theta(\tilde{\varphi}_\tau, L)$.

§1 The nonholomorphic φ_τ^*.

Set

$$\varphi_\tau(x) = x \cdot e(Q(x)\tau)$$

$$L_M = M\mathbb{Z}, \quad M \in 2\mathbb{N}, \; L_M^* = \frac{1}{M}\mathbb{Z}.$$

$$\rightsquigarrow \theta(\tau, M^2) := \Theta(\varphi_\tau, L_M)$$

wt $3/2$, ρ_{M^2} on $Mp_2(\mathbb{Z})$

$$\left[\left(\mathbb{Z}, \frac{M' x^2}{2} \right) \rightsquigarrow \text{Weil rep } \rho_{M'}, \; M' \in 2\mathbb{N} \right]$$

Recall: harmonic Maass forms are ξ_k-preimage of hol. mod. forms.

We want to construct hMf $\tilde{\theta}(\tau, M^2)$

s.t. $\xi_{1/2}(\tilde{\theta}(\tau, M^2)) = \theta(\tau, M^2)$.

Want to find $\tilde{\varphi}_\tau$ satisfying ①–③ and

④ $\xi_{1/2}(\tilde{\varphi}_\tau) = \varphi_\tau$.

<u>Def 1</u>:

$$\varphi_\tau^*(x) := e(-Q(x)\tau) \cdot \operatorname{sgn}(x) \cdot \operatorname{erfc}(\sqrt{2\pi v}\,|x|)$$

where $\operatorname{erfc}(y) := \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-r^2}\,dr$.

<u>Prop 1</u>: φ_τ^* satisfies ①, ③, ④ but not ②.

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§2 The holomorphic φ_τ^+.

Def 2:

$$\varphi_\tau^+(x) := e(-Q(x)\tau) \cdot \mathrm{sgn}(x) \cdot \left(\frac{-2i}{\eta^3(\tau)}\right) \cdot \sum_{\substack{m > |x| \\ m \in \mathbb{Z}+1/2}} (m - |x|) \cdot e(Q(m)\tau) e\left(\frac{m}{2}\right)$$

Prop 2: $\tilde{\varphi}_\tau := \varphi_\tau^+ - \varphi_\tau^*$ satisfies ①, ②, ③ (unif. for $\varepsilon \in (-1/2, 1/2)$), ④ and is in $L^1(\mathbb{R}) \cap C(\mathbb{R})$ and is differentiable on $\mathbb{R} \setminus (\mathbb{Z} + 1/2)$.

Thm 1

$$\tilde{\theta}(\tau, M^2) := M \cdot \lim_{\varepsilon \to 0} \sum_{h \in L_M^*/L_M} e_h \cdot \sum_{\lambda \in L_M + h + \varepsilon} \tilde{\varphi}_\tau(\lambda)$$

is a hMf of wt $1/2$, $\overline{\rho_{M^2}}$ on $Mp_2(\mathbb{Z})$

s.t. $\xi_{1/2}(\tilde{\theta}(\tau, M^2)) = \theta(\tau, M^2) := M \cdot \Theta(\varphi_\tau, L_M)$.

Furtheremore, the hol part $\theta^+(\tau, M^2)$ of $\tilde{\theta}(\tau, M^2)$ is in $\mathbb{Z}((q^{1/M}))[L_M^*/L_M]$.

Rmk: We can use $\tilde{\theta}(\tau, M^2)$ to construct $\tilde{\theta}(\tau, M)$, which maps $\theta(\tau, M) := \Theta(\varphi_\tau, K_M)$

$K_M = (\mathbb{Z}, \frac{Mx^2}{2})$.