Arakelov intersections on modular curves

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Notation

- $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ congruence subgroup
- v_Γ hyperbolic volume of a fundamental domain of Γ
- $X(\Gamma) = \overline{\Gamma \backslash \mathbb{H}}$ compact Riemann surface of genus $g_\Gamma \geq 2$
- $\mathcal{S}_2(\Gamma)$ cusp forms of weight 2 of Γ

Notation

Let $\{f_1, \ldots, f_{g_\Gamma}\}$ be an ONB of $\mathcal{S}_2(\Gamma)$. We define

$$F_\Gamma(z) := \frac{\mathrm{Im}(z)^2}{g_\Gamma} \sum_{j=1}^{g_\Gamma} |f_j(z)|^2$$

$$\mu_{\mathrm{can}}(z) := F_\Gamma(z)\, \mu_{\mathrm{hyp}}(z)$$

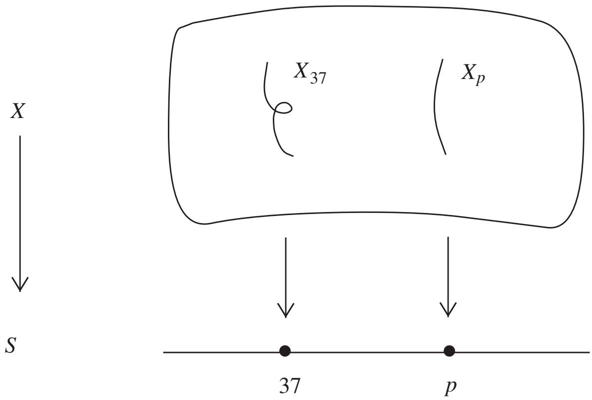
Denote by $g_{\mathrm{can}}(z, w)$ the canonical Green's function associated with $\mu_{\mathrm{can}}(z)$.

Fact 1: there exists a smooth algebraic curve X_Γ/K over a number field K such that

$$X_\Gamma(\mathbb{C}) \simeq X(\Gamma).$$

Fact 2: there exists an arithmetic surface $\mathscr{X}_\Gamma/\mathcal{O}_K$ such that $(\mathscr{X}_\Gamma)_\eta \simeq X_\Gamma$. This arithmetic surface is unique if it is minimal.

Pictures



Self-intersection of dualizing sheaf

On the arithmetic surface $\mathscr{X}_\Gamma/\mathcal{O}_K$ one has

- the *canonical bundle*: ω_Γ,
- Arakelov intersection theory.

It makes sense to talk about $\overline{\omega}_\Gamma^2$.

The case $\Gamma = \Gamma(N)$

For $N \geq 3$ odd and square-free integer

$$\frac{1}{\varphi(N)}\overline{\omega}_\Gamma^2 = \mathrm{geometric} + \mathrm{analytic}$$

Today, we will talk about

$$\mathrm{analytic} = 4g_\Gamma(g_\Gamma - 1)g_{\mathrm{can}}^\Gamma(0, \infty).$$

The case $\Gamma = \Gamma(N)$

Theorem

For $N \geq 3$ odd and square-free, we have

$$4g_\Gamma(g_\Gamma - 1)g_{\mathrm{can}}^\Gamma(0,\infty) - \kappa_N = 4g_\Gamma \log(N) + o(g_\Gamma \log(N)),$$

where

$$\kappa_N = \frac{4\pi^2 g_\Gamma}{N^2\varphi(N)}\left(1 - \frac{6}{N}\right)\sum_{\substack{\chi\neq\chi_0\\ \text{even}}} \frac{L(1,\chi)}{L(2,\chi)}.$$

Previous results

Mayer (2012): $\Gamma = \Gamma_1(N)$

$$4g_\Gamma(g_\Gamma - 1)g^\Gamma_{\mathrm{can}}(0,\infty) - \kappa^*_N = 2g_\Gamma \log(N) + o(g_\Gamma \log(N)).$$

Abbes–Ullmo (1997): $\Gamma = \Gamma_0(N)$

$$4g_\Gamma(g_\Gamma - 1)g^\Gamma_{\mathrm{can}}(0,\infty) = 2g_\Gamma \log(N) + o(g_\Gamma \log(N)).$$

Furthermore,

$$\overline{\omega}^2_\Gamma = 3g_\Gamma \log(N) + o(g_\Gamma \log(N)).$$

Spectral interpretation of $g_{\mathrm{can}}(z, w)$

$$g^{\Gamma}_{\mathrm{can}}(0,\infty) = -2\pi\mathscr{C}^{\Gamma}_{0\infty} - \frac{2\pi}{v_\Gamma} + 4\pi\mathscr{R}^{\Gamma}_{\infty} + 2\pi \lim_{s\to 1}\left(\frac{v_\Gamma^{-1}}{s(s-1)} + \int_{X(\Gamma)\times X(\Gamma)} G^{\Gamma}_{s}(z,w)\mu_{\mathrm{can}}(z)\mu_{\mathrm{can}}(w)\right).$$

- $\mathscr{C}^{\Gamma}_{0\infty}$: scattering constant.
- $\mathscr{R}^{\Gamma}_{\infty}$: constant term in the Laurent expansion of Rankin–Selberg transform of F_Γ at $s = 1$.
- $G^{\Gamma}_{s}(z, w)$: automorphic Green's function.

Scattering constants

$\Gamma = \Gamma_0(N)$

$$\mathscr{C}_{\infty 0}^{\Gamma_0(N)} = 2v_{\Gamma_0(N)}^{-1} \times (\text{factor}),$$

$\Gamma = \Gamma(N)$

$$\mathscr{C}_{\infty 0}^{\Gamma(N)} = 2v_{\Gamma(N)}^{-1}\left(\text{factor} + \kappa_N - \frac{1}{2}\log(N)\right),$$

The constant $\mathscr{R}_\infty^\Gamma$

Main tools:

- spectral expansion of automorphic kernels of weight 0 and 2
- In the hyperbolic contribution, we obtain identities of the type

$$\zeta_\Gamma(s) = (\text{factor}) \times \zeta(s, \mathfrak{c}),$$

where $\mathfrak{c}$ is a narrow-ray ideal class. E.g., for $\Gamma = \Gamma(N)$ we have

$$\zeta_\Gamma(s) = \frac{1}{N^s}\zeta(s, \mathfrak{c}).$$