Traces of CM values and geodesic cycle integrals of harmonic Maass forms

○ Zagier's results on the traces of singular moduli

$J(z) = j(z) - 744 \in M_0^!(SL_2(\mathbb{Z}))$

$-d < 0 \quad \Delta > 0$ fund discr.

- $\alpha_Q \in \mathbb{H}$ CM point, i.e. zero of an integral bin. quadr. form $Q(x,y) = ax^2 + bxy + cy^2$ (with $-d\Delta = b^2 - 4ac$)

Thm The gen. series of the twisted traces of CM values of J

$$t_\Delta(J; d) = \frac{1}{\sqrt{\Delta}} \sum_{Q \in SL_2(\mathbb{Z}) \backslash \mathcal{Q}_{-d\Delta}} \chi_\Delta(Q)\, J(\alpha_Q)$$

are weakly hol. forms of wt 1/2 and 3/2.

$$f_d(z) = q^{-d} + \sum_{\Delta > 0} t_\Delta(J; d)\, q^\Delta \in M^!_{1/2}(\Gamma_0(4))$$

$$g_\Delta(z) = q^{-\Delta} + \sum_{d \geq 0} t_\Delta(J; d)\, q^d \in M^!_{3/2}(\Gamma_0(4))$$

Prelims

general situation

V = rational quadr. space with a non-deg. bil. form $(\,,)$ of signature (b^+, b^-)

$Q(X) = \frac{1}{2}(X,X)$ the ass. quadr form

$L \subset V(\mathbb{Q})$ an even lattice of full rank

L' = dual lattice of L

L'/L = discr. gp (finite + abelian)

$G = \mathrm{Spin}(V)$

K a max'l cpct subgp of $G(\mathbb{R})$

$D = G(\mathbb{R})/K$ the ass. symmetr. space

Γ = congr. subgp of $SL_2(\mathbb{Q})$ that takes L to itself and acts triv. on L'/L

"Example"

$V = \left\{ X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in \mathbb{Q}^{2\times 2} \right\}$

with $(X,Y) = -\underset{\geq 1}{N}\, \mathrm{tr}(X\cdot Y)$

$Q(X) = N \cdot \det(X)$

(V,Q) is a quadr. space over $\mathbb{Q}$ of sign. $(1,2)$

$L = \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} : a,b,c \in \mathbb{Z} \right\}$

$L' = \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} : a,b,c \in \mathbb{Z} \right\}$

$L'/L \cong \mathbb{Z}/2N\mathbb{Z}$

$G \simeq SL_2$

$K \simeq SO(2)$

$D \simeq \mathbb{H}$

$\simeq \{ z \subset V(\mathbb{R});\ \dim z = 1,\ \text{and}\ (\,,)|_z > 0 \}$

$\Gamma = \Gamma_0(N)$

$\left(= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0\ (N) \right\}\right)$

$z = x + iy \in D\ (\simeq \mathbb{H})$ with $x, y \in \mathbb{R}$

$M = \Gamma \backslash D$ $\qquad\qquad$ $\Gamma_0(N)\backslash \mathbb{H}$

the (symplectic) symmetric space associated to $SL_2 \simeq Sp_2$
$\cong \mathbb{H}$

$\tau = u + iv$, $u, v \in \mathbb{R}$
$\in \mathbb{H}$

Sp_2 and $O(1,2)$ form a dual reductive pair in the sense of Howe

$\Rightarrow$ we can lift autom for Sp_2 to autom. form for $O(1,2)$ (and vice versa)

$Mp_2(\mathbb{R})$ = metaplectic group

$= \{ (\gamma, \phi) : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ hol s.t. $\phi(\tau)^2 = c\tau + d \}$

($\sqrt{w}$ = pr. branch of the square root)

$Mp_2(\mathbb{Z})$ = inverse image of $SL_2(\mathbb{Z})$ under

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right)$$

$$= \Big\langle \underbrace{\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right)}_{=: T}, \underbrace{\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right)}_{=: S} \Big\rangle$$

$\triangleright$ $\mathbb{C}[L'/L]$ = gp algebra

$$= \Big\{ \sum_{h \in L'/L} a_h \underset{\substack{\uparrow \\ \text{standard basis vectors of } L'/L}}{e_h} : a_h \in \mathbb{C} \Big\}$$

There is a unitary repr ρ_L of $Mp_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ which is defined by the action on T and S via

$$\rho_L(T)\, e_h = \exp(2\pi i Q(h))\, e_h$$

$$\rho_L(S)\, e_h = \frac{\sqrt{i}^{\,-b^+ + b^-}}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} \exp(-2\pi i\, (h,h'))\, e_{h'}$$

ρ_L = Weil repr. attached to L

<u>Def'n</u> A smooth fct $f: \mathbb{H} \to \mathbb{C}[L'/L]$ is called <u>harmonic Maass form of wt k $(\in \frac{1}{2}\mathbb{Z})$ w.r.t. the repr. ρ_L and the gp $Mp_2(\mathbb{Z})$</u> if:

(1) $f(\gamma\tau) = \phi(\tau)^{2k} \rho_L(\gamma,\phi) \cdot f(\tau) \;\; \forall\, (\gamma,\phi) \in Mp_2(\mathbb{Z})$

(2) $\Delta_k f = 0$ where $\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$
$(\tau = x+iy)$

(3) f has at most linear exponential growth at $i\infty$

H_{k,ρ_L}

Rk: $k \neq 1$,

F^+ hol part

$$F(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c_F^+(n,h) q^n e_h$$

$$+ \underbrace{\sum_{h \in L'/L} \Big(c_F^-(0,h) y^{1-k} + \sum_{\substack{n \in \mathbb{Q}\setminus\{0\} \\ n \ll \infty}} c_F^-(n,h) \underbrace{H(2\pi n y)}_{H(w) = e^{-w} \int_{-2w}^{\infty} e^{-t} t^{-k} dt} e^{2\pi i x n} \Big) e_h}_{F^- \text{ non-hol part}}$$

Let $\xi_k = 2i y^k \overline{\frac{\partial}{\partial \bar z}}$

$$\xi_k : H_{k,\rho_L} \longrightarrow \underbrace{M^!_{2-k,\bar\rho_L}}_{\substack{(2)\ \text{hol. on } \mathbb{H} \\ (1)\ \checkmark \\ (3)\ \text{poles at } i\infty}}$$

$$H^+_{k,\rho_L} = \xi_k^{-1}\left(S_{2-k,\bar\rho_L} \right)$$

Heegner divisors and geodesics

- $X \in V$ with $Q(X) = m \in \mathbb{Q}_{>0}$.

  $D_X = \mathrm{span}(X) \in D \, (\cong \mathbb{H})$ the Heegner pt of discr. m ass. to X

  Define $$t(\underset{\substack{\uparrow \\ \Gamma\text{-inv.}}}{F}; m, h) = \sum_{X \in \Gamma \backslash \underbrace{L_{m,h}}_{=\{X \in L+h \mid Q(X) = m\}}} \frac{1}{|\bar{\Gamma}_X|} \overset{\substack{\text{image of } D_X \text{ in } M \\ \swarrow}}{F(D_X)}$$

- $X \in V$ with $Q(X) = m \in \mathbb{Q}_{<0}$

  $c_X = \{ z \in D : z \perp X \}$

  $(= \{ z \in D : c|z|^2 - b\,\mathrm{Re}(z) + a^{\$} = 0 \})$

  $c(X) = \Gamma_X \backslash c_X$

  <u>case 1</u> $\frac{|m|}{N}$ is not a square

  $\Leftrightarrow$ $\bar{\Gamma}_X$ is ∞ cyclic

  $\Leftrightarrow$ $c(X)$ is finite

  <u>case 2</u> $\frac{|m|}{N}$ is a square

  $\Leftrightarrow$ $\bar{\Gamma}_X$ is triv.

  $\Leftrightarrow$ $c(X)$ is inf.

Recent results

Bruinier/Funke

take $F \in H_0^+(N)$ ← for $\Gamma_0(N)$
$\underbrace{}_{\text{scalar-valued}}$

consider: $$I^{KM}(\tau, F) = \int_M F(z) \underbrace{\Theta_L(\tau, z, \varphi_{KM})}\, d\mu(z)$$

- wt 0 in z
- wt 3/2 in τ
- square exponentially decr towards the cusps

"Kudla-Millson theta fct"

Thm $I^{KM}(\tau, F) \in H_{3/2, \rho_L}$ and the coeff of index (m,h) for m>0 is given by: ← the hol part of

$$t(F; m, h)$$

⟶ recover g_1

Br/Ono: take $F \in H^+_{-2k}(N)$
$$R_k = 2i \frac{\partial}{\partial z} + k y^{-1} \quad (\text{wt } k \xrightarrow{R_k} \text{wt } k+2)$$

$k \in 2\mathbb{N}+1$: $\underset{\text{lowr op.}}{L_{3/2, \tau}^{k+1/2}} \int_M R^k_{-2k} F(z) \cdot \Theta_L(\tau, z, \varphi_{KM}) \in H^+_{1/2-k, \rho_L}$

$k \in 2\mathbb{N}$: $R^{k/2}_{3/2, \tau} \int$ " $\in H^+_{3/2+k, \rho_L}$

and again coeff of the hol part are given by the traces

⟶ formula for the partition fct