Cyclic Orbifold Construction of Holomorphic Vertex Operator Algebras

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Introduction

Obtain holomorphic VOA $\widetilde{V}$ from another holomorphic VOA V:

Orbifolding

$$V \xrightarrow[\supseteq V^{\sigma} \subseteq]{\sigma \in \mathrm{Aut}(V)} \widetilde{V}$$

Development:

- [FLM88]: $V^{\natural}$ as first example of $\mathbb{Z}_2$-orbifold construction,
- [DGM90,. . .]: further $\mathbb{Z}_2$-orbifold constructions,
- [Miy13]: $\mathbb{Z}_3$-orbifold construction,
- [EMS]: general $\mathbb{Z}_n$-orbifold theory

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Vertex Operator Algebras

Definition (Vertex Operator Algebra, Data)

- (*space of states*) $\mathbb{Z}$-graded $\mathbb{C}$-vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ with $V_n = 0$ for $n \ll 0$ and $\dim(V_n) < \infty$ for all $n \in \mathbb{Z}$,
- (*vacuum vector*) non-zero vector $\mathbf{1} \in V_0$,
- (*conformal vector*) non-zero vector $\omega \in V_2$,
- (*translation operator*) linear operator $T : V \to V$ of weight 1,
- (*vertex operators*) linear map

$$Y(\cdot, z) : V \to \operatorname{End}(V)[[z^{\pm 1}]], \quad Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$

with $a_n b = 0$ for $n \gg 0$ and $\operatorname{wt}(a_n) = \operatorname{wt}(a) - n - 1$, $a, b \in V$

Vertex Operator Algebras

Definition (Vertex Operator Algebra, Axioms)

- (*vacuum axiom*) $Y(\mathbf{1},z)=\mathrm{id}_V$ and $Y(a,z)\mathbf{1}|_{z=0}=a$, $a\in V$.
- (*translation axiom*) $[T,Y(a,z)]=\partial_z Y(a,z)$ and $T\mathbf{1}=0$.
- (*locality axiom*) for $a,b\in V$ there is $N\gg 0$ s.t.

$$(z-w)^N[Y(a,z),Y(b,w)]=0 \quad \in \mathrm{End}(V)[[z^{\pm 1},w^{\pm 1}]].$$

- (*Virasoro relations*) $Y(\omega,z)=:\sum_{n\in\mathbb{Z}}L_n z^{-n-2}$ satisfies

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}c$$

for some *central charge* $c\in\mathbb{C}$. In addition, $L_{-1}=T$ and $L_0a=na=\mathrm{wt}(a)a$, $a\in V_n$.

Examples of VOAs

Example (The Moonshine Module)

Frenkel, Lepowsky, Meurman constructed the VOA $V^\natural$ of central charge 24 whose automorphism group is the Monster, i.e. $\operatorname{Aut}(V^\natural) \cong \mathbb{M}$. Used in Borcherds' proof of the Moonshine conjecture. The VOA $V^\natural$ is "nice" and holomorphic.

Example (Lattice VOAs)

Let L be a positive-definite, even lattice. We can associate with it a VOA V_L of central charge $c = \operatorname{rk}(L)$. For any L the VOA V_L is "nice". The isomorphism classes of irreducible modules of V_L can be parametrised by the elements of L'/L. In particular, if L is unimodular, then V_L is holomorphic.

Modules for VOAs

Definition (VOA Module)

Let V be a VOA. A V-module W is:

- $\mathbb{C}$-graded $\mathbb{C}$-vector space $W = \bigoplus_{\lambda\in\mathbb{C}} W_\lambda$ with $W_\lambda = 0$ for $\mathrm{Re}(\lambda) \ll 0$ and $\dim(V_\lambda) < \infty$ for all $\lambda \in \mathbb{C}$,
- equipped with linear map

$$Y_W(\cdot, z) : V \to \mathrm{End}(W)[[z^{\pm 1}]], \quad Y_W(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$

  with $a_n b = 0$ for $n \gg 0$
- s.t. *all the defining properties of a VOA that make sense hold*
- (but replace locality axiom by Jacobi identity).

"Nice" Vertex Operator Algebras

Definition ("Niceness")

A VOA V is called "nice" if it is

- *rational*, i.e. the category of V-modules is semisimple with finitely many irreducible objects (up to isomorphism),
- *C_2-cofinite*, i.e. $\dim(V/\{v_{-2}w | v, w \in V\}) < \infty$,
- *simple*, i.e. V has no non-trivial ideal,
- of *CFT-type*, i.e. $V = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} V_n$ and $\dim(V_0) = 1$,
- *self-contragredient*, i.e. $V \cong V'$ (as modules).

Definition (Holomorphicity)

A VOA V is called *holomorphic* if V is rational with exactly one irreducible module (up to isomorphism).

Examples of VOAs, revisited

Example (The Moonshine Module)

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Fusion Rules

Theory of *fusion products* (tensor products) of modules for VOAs (Huang-Lepowsky, Li):

- VOA V rational: fusion product for V-modules exists.
- *Fusion algebra*: let V have irred. modules $W^1,\ldots,W^r$:

$$W^i \boxtimes_V W^j \cong \bigoplus_{k=1}^{r} \underbrace{(W^k \oplus \ldots \oplus W^k)}_{N_{ij}^k \text{ times}} = \bigoplus_{k=1}^{r} N_{ij}^k W^k.$$

Theorem

Let V be a "nice" VOA. Then the associated fusion algebra $\mathcal{V}(V)$ is a finite-dimensional, commutative, associative, unital $\mathbb{C}$-algebra (with unit V).

- Simple-current case: $\mathcal{V}(V) = \mathbb{C}[E]$ with fusion group E.

Zhu's Modular Invariance

[Zhu96] Consider "nice" VOA V with irreducible modules $W^1, \ldots, W^r$:

- Have holomorphic trace functions

$$T_{W^i}(v, \tau) := \operatorname{tr}|_{W^i} o(v) q^{L_0 - c/24},$$

$o(v) = v_{\mathrm{wt}(v)-1}$, $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$, $\tau \in \mathbb{H}$.

- Modular invariance: let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$: there is a complex $(r \times r)$-matrix $M(\gamma) = (m_{i,j})_{i,j=1}^r$ s.t.

$$\frac{1}{(c\tau + d)^k} T_{W^i}\left(v, \frac{a\tau + b}{c\tau + d}\right) = \sum_{j=1}^r m_{i,j} T_{W^j}(v, \tau)$$

for all $k \in \mathbb{Z}_{\geq 0}$, $\tau \in \mathbb{H}$, $v \in V_{[k]}$.

- Important case: $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \rightsquigarrow S$-Matrix.

Verlinde Formula

Theorem (Verlinde, Huang)

Let V be a "nice" VOA. Let $W^1 = V, W^2, \ldots, W^r$ be the irreducible modules of V. Then S is symmetric and the square S^2 is a permutation matrix which shifts i to i' where $W^{i'} := (W^i)'$. Moreover, we have the following formula

$$N_{i,j}^k = \sum_{l=1}^{r} \frac{S_{il} S_{jl} S_{lk'}}{S_{1l}} \qquad \text{(Verlinde formula)}$$

for the fusion rules $N_{i,j}^k$ of V.

- In simple-current situation: combine Verlinde formula and $STS = T^{-1}ST^{-1}$ from modular invariance to compute S- and T-matrix of the VOA (and hence the fusion rules).

Automorphisms of VOAs

Definition (VOA Automorphism)

Let V be a VOA. A *VOA automorphism* g of V is an automorphism of the $\mathbb{C}$-vector space V such that

- $gY(a,z)g^{-1} = Y(ga,z)$ for $a \in V$,
- $g\omega = \omega$ (g is grade-preserving),
- $g\mathbf{1} = \mathbf{1}$.

- decompose V into eigenspaces

$$V = \bigoplus_{r\in\mathbb{Z}_n} V^r = \{a \in V | ga = \mathrm{e}^{(2\pi\mathrm{i})r/n} a\}$$

with $n = \mathrm{ord}(g)$
- fixed-point subVOA: $V^g = V^0$

Properties of Fixed-Point SubVOAs

Inherited properties of V^G from V:

Proposition

- *Let V be a simple, self-contragredient VOA of CFT-type and let G be a finite group of automorphisms of V. Then the fixed-point subVOA V^G is again simple [DM97], self-contragredient and of CFT-type.*
- *If in addition V is C_2-cofinite and G is solvable, then V^G is also C_2-cofinite [Miy13].*
- *If in addition (to both points) V is rational and G is cyclic, then V^G is also rational [Miy10].*

- In total: If V is "nice", then so is V^g for $g \in \mathrm{Aut}(V)$.

Twisted Modules for VOAs

Definition (Twisted VOA Module)

g-twisted module W for a VOA V: like untwisted module but

- linear map

$$Y_W(\cdot,z): V \to \operatorname{End}(W)[[z^{\pm 1/n}]], \quad Y_W(a,z) = \sum_{k\in\mathbb{Q}} a_k z^{-k-1}$$

  with $Y_W(a,z) = \sum_{k\in -r/n+\mathbb{Z}} a_k z^{-k-1}$ for $a \in V^r$,
- replace Jacobi identity by "twisted version".

- A g-twisted V-module is an untwisted (ordinary) V^g-module.
- Let V be holomorphic. Then V posesses a unique (up to isomorphism) irreducible g-twisted module, call it $V(g)$ [DLM00].

Automorphism of the Twisted Modules

Proposition

Let V be a "nice" holomorphic VOA and let $G = \langle\sigma\rangle$ be a cyclic group of automorphisms of V of order n. Then for each $h \in G$ there is a unique (up to root of unity) representation

$$\phi_h : G \to \mathrm{Aut}_{\mathbb{C}}(V(h))$$

of G on the vector space $V(h)$ such that

$$\phi_h(g) Y_{V(h)}(v, z) \phi_h^{-1}(g) = Y_{V(h)}(gv, z)$$

for all $g \in G$ and $v \in V$.

- Clearly one can take $\phi_e(g) = g$ on $V = V(e)$.

Modules for the Fixed-Point SubVOA

Theorem (Classification of Irreducible Modules [Miy10,...])

Let V be a "nice" holomorphic VOA and let $G = \langle\sigma\rangle$ be a finite, cyclic group of automorphisms of V of order n. Then, every V^G-module is completely reducible and, up to isomorphism, there are exactly n^2 distinct irreducible V^G-modules, namely $W^{(i,j)}$, $i, j \in \mathbb{Z}_n$. $W^{(i,j)}$ is the eigenspace in $V(\sigma^i)$ of $\phi_{\sigma^i}(\sigma)$ corresponding to the eigenvalue ξ_n^j.

Next steps:

- Show that all the modules are simple currents, i.e. we have group-like fusion.
- Use twisted version of Zhu's modular invariance [DLM00].
- Compute the S- and T-matrix and fusion rules of V^σ.

Main Result

Theorem (Main Result I [EMS])

Let V be a "nice" holomorphic VOA and let σ be an automorphism of V of order n. Then the fusion algebra of V^g is the group algebra $\mathbb{C}[E]$ of a group E, which is a central extension

$$1 \leftarrow \mathbb{Z}_n \leftarrow E \leftarrow \mathbb{Z}_n \leftarrow 1,$$

i.e. we have

$$W^{(i,j)} \boxtimes W^{(l,k)} \cong W^{(i+l,j+k+c(i,l))}$$

for some 2-cocycle $c \in Z^2(\mathbb{Z}_n, \mathbb{Z}_n)$.

- The class $r \in \mathbb{Z}_n \cong H^2(\mathbb{Z}_n, \mathbb{Z}_n)$ of the 2-cocycle is known. We say σ is of type $n\{r\}$.

Main Result

Theorem (Main Result II [EMS])

Let V as in the above theorem and of type $n\{0\}$. The direct sum of irreducible V^σ-modules

$$\widetilde{V} := \bigoplus_{i \in \mathbb{Z}_n} W^{(i,0)}$$

admits the structure of a "nice" holomorphic VOA extending the VOA structure of V^σ.

- The sum $V = \bigoplus_{j \in \mathbb{Z}_n} W^{(0,j)}$ gives back the original VOA V.
- Uses deep sitting results by Huang, Dong, Lepowsky,... on abelian intertwining algebras, modular tensor categories,...
- Closely related to "cohomology of abelian groups" by Eilenberg and Mac Lane.

Schellekens' List (Existence)

Proposition ([Zhu96])

Let V be a "nice" holomorphic VOA. Then the central charge of V is a positive multiple of 8.

Theorem ([Sch93, EMS])

Let V be a "nice" holomorphic VOA of central charge 24. Then the Lie algebra structure of V_1 has to be isomorphic to one of the 71 Lie algebras on Schellekens' list.

Conjecture

The Lie algebra structure determines the VOA uniquely (up to isomorphism), i.e. there are at most 71 "nice" holomorphic VOAs of central charge 24.

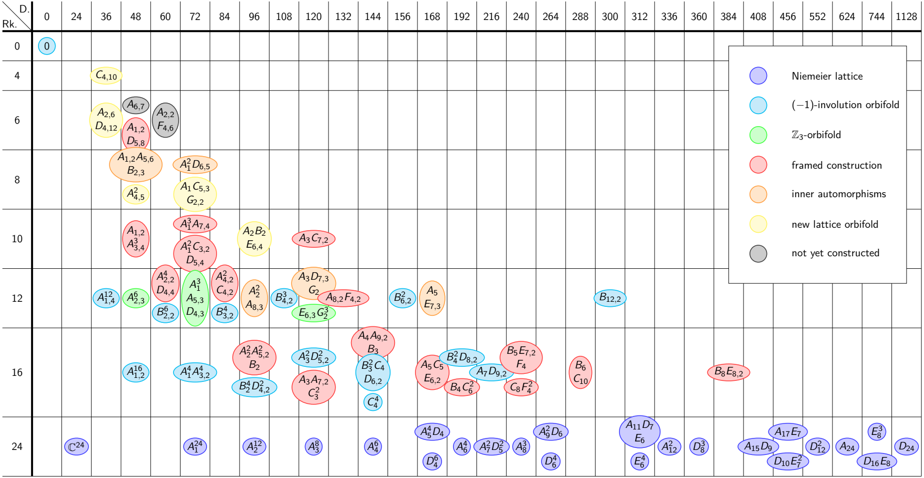
Schellekens' List (Construction)

Construction of 69 of the 71 VOAs in Schellekens' list:

- 24: lattice VOAs associated to the 24 Niemeier lattices [Bor86, FLM88, Don93],
- 15: $\mathbb{Z}_2$-orbifolds for V_L and $-1 \in \mathrm{Aut}(L)$ [FLM88, DGM96],
- 17: repeated $\mathbb{Z}_2$-orbifolds (framed VOAs) [Lam11, LS12],
- 3: $\mathbb{Z}_3$-orbifolds for V_L [Miy13, SS13],
- 5: $\mathbb{Z}_2$-orbifolds using inner automorphisms [LS15],
- 5: $\mathbb{Z}_n$-orbifolds for V_L ($n = 4, 5, 6, 10$) [EMS]

Conjecture

Every of the 71 VOAs in Schellekens' list exists.



D.
Rk.
0 24 36 48 60 72 84 96 108 120 132 144 156 168 192 216 240 264 288 300 312 336 360 384 408 456 552 624 744 1128
0 4 6 8 10 12 16 24
Niemeier lattice
(−1)-involution orbifold
$\mathbb{Z}_3$-orbifold
framed construction
inner automorphisms
new lattice orbifold
not yet constructed
0
$C_{4,10}$
$A_{2,6}$ $D_{4,12}$
$A_{6,7}$
$A_{1,2}$ $D_{5,8}$
$A_{2,2}$ $F_{4,6}$
$A_{1,2}A_{5,6}$ $B_{2,3}$
$A_{4,5}^2$
$A_1^2 D_{6,5}$
$A_1 C_{5,3}$ $G_{2,2}$
$A_{1,2}$ $A_{3,4}^3$
$A_1^3 A_{7,4}$
$A_1^2 C_{3,2}$ $D_{5,4}$
$A_2 B_2$ $E_{6,4}$
$A_3 C_{7,2}$
$A_{1,4}^{12}$
$A_{2,3}^6$
$A_{2,2}^4$ $D_{4,4}$
$B_{2,2}^6$
A_1^3 $A_{5,3}$ $D_{4,3}$
$A_{4,2}^2$ $C_{4,2}$
$B_{3,2}^4$
A_2^2 $A_{8,3}$
$B_{4,2}^3$
$A_3 D_{7,3}$ G_2
$A_{8,2} F_{4,2}$
$E_{6,3} G_2^3$
$B_{6,2}^2$
A_5 $E_{7,3}$
$B_{12,2}$
$A_{1,2}^{16}$
$A_1^4 A_{3,2}^4$
$A_2^2 A_{5,2}^2$ B_2
$B_2^4 D_{4,2}^2$
$A_3^2 D_{5,2}^2$
$A_3 A_{7,2}$ C_3^2
$A_4 A_{9,2}$ B_3
$B_3^2 C_4$ $D_{6,2}$
C_4^4
$A_5 C_5$ $E_{6,2}$
$B_4^2 D_{8,2}$
$B_4 C_6^2$
$A_7 D_{9,2}$
$B_5 E_{7,2}$ F_4
$C_8 F_4^2$
B_6 C_{10}
$B_8 E_{8,2}$
$\mathbb{C}^{24}$
A_1^{24}
A_2^{12}
A_3^8
A_4^6
$A_5^4 D_4$
D_4^6
A_6^4
$A_7^2 D_5^2$
A_8^3
$A_9^2 D_6$
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E_6^4
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D_8^3
$A_{15} D_9$
$A_{17} E_7$
$D_{10} E_7^2$
D_{12}^2
A_{24}
E_8^3
$D_{16} E_8$
D_{24}

Thank you for your attention!