Cutting out arithmetic Teichmüller curves in genus 2 with theta functions

Joint work in progress with Martin Möller

André Kappes

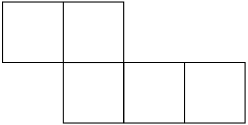
Goethe-Universität Frankfurt

March 2, 2015

Square-tiled surfaces

Glue d unit squares.

$$d = 5$$



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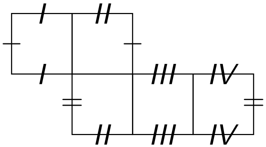

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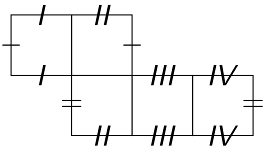

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How many possibilities?

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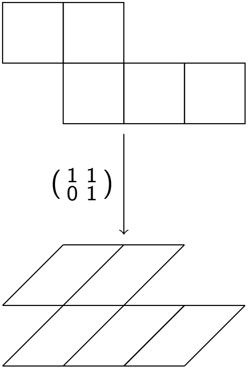

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$\mathrm{SL}_2(\mathbb{Z})$ acts on the set of square-tiled surfaces.

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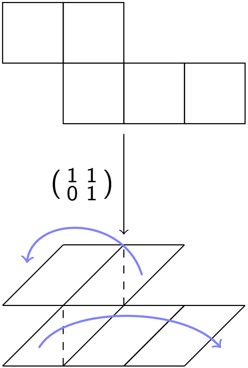

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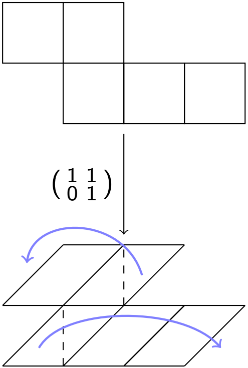

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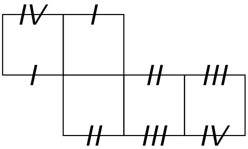

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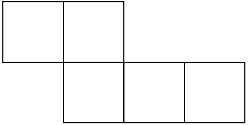
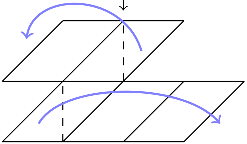
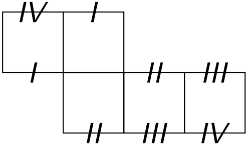


$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



$=$



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More involved question

How many $\mathrm{SL}_2(\mathbb{Z})$-orbits? Sizes?

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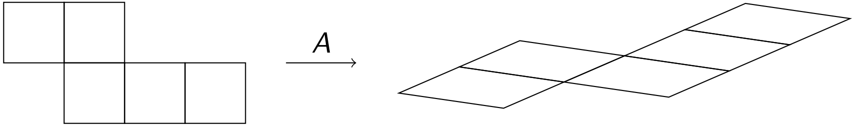
Let $A \in \mathrm{SL}_2(\mathbb{R})$.



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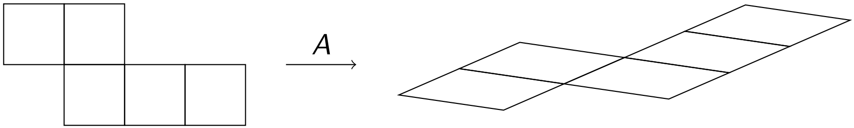

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$$\mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathcal{M}_g$$

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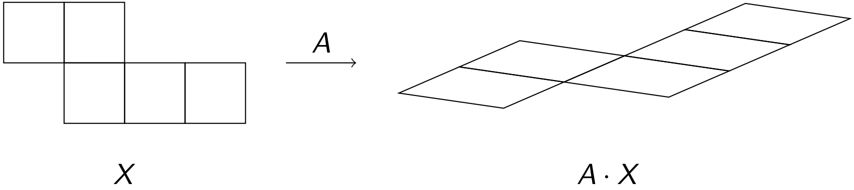

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$$\mathbb{H} = \mathrm{SO}(2)\backslash\,\mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathcal{M}_g$$

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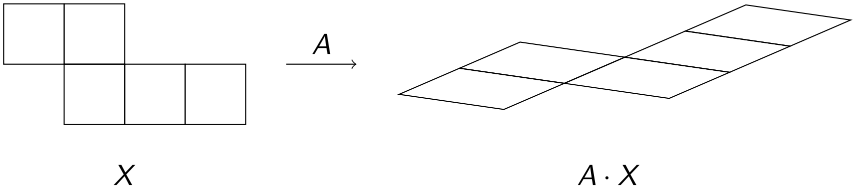
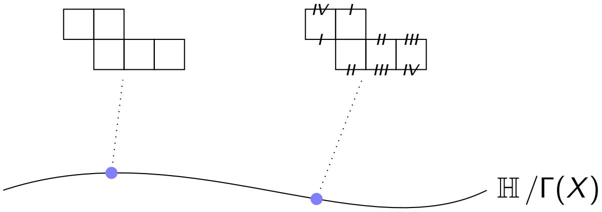


Defines a map

$$\mathbb{H}/\Gamma(X) \longrightarrow \mathcal{M}_g$$

$\Gamma(X) = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(X)$

Arithmetic Teichmüller curves

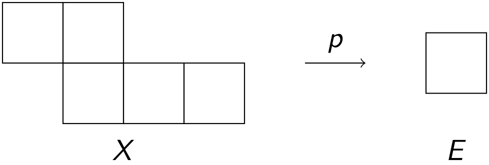


More involved question'

How many arithmetic Teichmüller curves?
What are their Euler characteristics?

Note: size($\mathrm{SL}_2(\mathbb{Z})$-orbit) $= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(X)] = -6 \cdot \chi(\mathbb{H}/\Gamma(X))$.

Square-tiled surfaces are torus covers



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$+$ covering $p : X \to E,\ \ g(E) = 1$
s. th. p ramified over at most 1 point.

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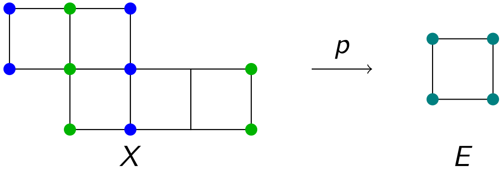

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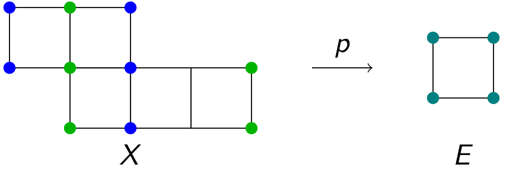

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Definition

A square-tiled surface $p : X \to E$ is called **primitive**, if there are no intermediate covers

Genus 2, one double ramification point

- Complete classification of $\mathrm{SL}_2(\mathbb{Z})$-orbits / arithmetic Teichmüller curves of primitive d-square tiled surfaces [McMullen, Hubert-Lelièvre]

$W_{d^2,\varepsilon}$ Weierstraß curves

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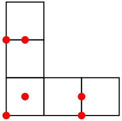
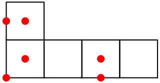


$\varepsilon = 3$



$\varepsilon = 1$

Invariants

degree $d = \deg(p : X \to E)$

spin $\varepsilon = \#\{\text{integral Weierstraß points}\}$

$$\varepsilon \in \begin{cases} 1 \text{ or } 3, & d \equiv 1 \bmod 2 \\ 2, & d \equiv 0 \bmod 2 \end{cases}$$

Genus 2, one double ramification point

- Complete classification of $\mathrm{SL}_2(\mathbb{Z})$-orbits / arithmetic Teichmüller curves of primitive d-square tiled surfaces [McMullen, Hubert-Lelièvre]
- Euler characteristics / sizes of $\mathrm{SL}_2(\mathbb{Z})$-orbits [Bainbridge, Lelièvre-Royer]

Genus 2, two simple ramification points

Conjecture (Zmiaikou)

For $d \geq 7$ there exist 2 orbits of primitive d-square-tiled surfaces of genus 2 with 2 simple ramification points.

The associated Teichmüller curves $T_{d,\varepsilon}$ satisfy

$$\chi(T_{d,\varepsilon}) = \begin{cases} -\frac{1}{144}(d^2 - 8d + 15)\frac{\#\,\mathrm{SL}_2(\mathbb{Z}/d\,\mathbb{Z})}{d}, & \varepsilon = 3 \\ -\frac{1}{48}(d^2 - 4d + 3)\frac{\#\,\mathrm{SL}_2(\mathbb{Z}/d\,\mathbb{Z})}{d}, & \varepsilon = 1 \end{cases}$$

(here: case d odd)

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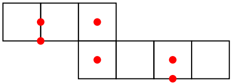

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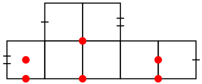

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Theorem (Möller-K)

The counting part of Zmiaikou's conjecture holds.

The Jacobian

X a compact Riemann surface, $g = 2$

Definition (Jacobian)

$J(X) = \mathbb{C}^2 / \Pi\, \mathbb{Z}^4, \qquad \Pi = (\int_{\gamma_j} \omega_i)$

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Theorem

The Torelli map

$$\mathcal{M}_2 \to \mathcal{A}_2, \qquad [X] \mapsto [J(X)]$$

is an embedding.

$\rightsquigarrow$ cut out Teichmüller curve in $\mathcal{A}_2$

Multiplication by $\mathfrak{o}_{d^2}$

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$\Leftrightarrow$ $\mathfrak{o}_{d^2} = \{(a, b) \in \mathbb{Z}^2 \mid a \equiv b \bmod d\} \subset \mathrm{End}(J(X))$

"$J(X)$ has multiplication by $\mathfrak{o}_{d^2}$"

Pseudo-Hilbert modular surfaces

Definition

X_{d^2} = moduli space of p.p. abelian surfaces with mult. by $\mathfrak{o}_{d^2}$
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- $X_{d^2} = \mathbb{H}^2/\Gamma_{d^2}$, where

$$\Gamma(d) \times \Gamma(d) \;\subset\; \Gamma_{d^2} \;\subset\; \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$$

$\Rightarrow$ X_{d^2} is sandwiched

$$X(d) \times X(d) \longrightarrow X_{d^2} \longrightarrow X(1) \times X(1)$$

The universal family

$$
\begin{array}{ccc}
A_{d^2} & = \mathbb{H}^2 \times \mathbb{C}^2 \,/\text{semidirect product} & \pi^{-1}(x) = A \\
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Find a subset $\tilde{C} \subset A_{d^2}$ such that $\pi(\tilde{C}) = T_{d,\varepsilon}$.

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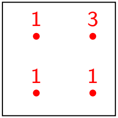
- Configuration of Weierstraß points



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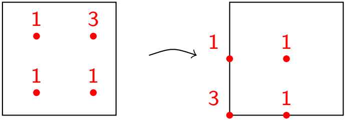

- wlog p is normalized: 3 Weierstraß points over 0.

Theta functions

Definition (Classical theta function)

$$\vartheta : \mathbb{H}_2 \times \mathbb{C}^2 \to \mathbb{C}, \quad (Z, u) \mapsto \sum_{x \in \mathbb{Z}^2} e^{\pi i (x^T Z x) + 2\pi i x^T u}$$

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In $g = 2$, the image of the Abel-Jacobi map is the zero locus of ϑ

$$\Phi(X) = \{\vartheta = 0\}$$

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$$0 = \frac{\partial}{\partial x}\vartheta(\Phi(x_0)) = \frac{\partial \vartheta}{\partial u_1}(\Phi(x_0)) \cdot \omega(x_0) + \frac{\partial \vartheta}{\partial u_2}(\Phi(x_0)) \cdot \omega_2(x_0)$$

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3. $p(x_0) = \begin{cases} 0 \in E & \Leftrightarrow p_*(u_0) = 0 \\ \text{2-torsion point } \neq 0 & \Leftrightarrow p_*(u_0) \text{ has order 2} \end{cases}$

Cutting out ramification point in the universal family

$$\begin{array}{ll} A_{d^2} & = \mathbb{H}^2 \times \mathbb{C}^2 \,/\text{semidirect product} \\ \pi \downarrow & \\ X_{d^2} & \end{array}$$

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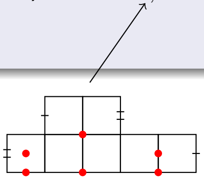

union of Teichm. curves

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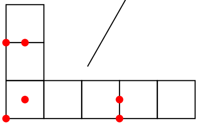

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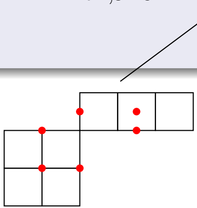

Push forward is section of Hodge bundle

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Corollary

For odd d*, the counting part of Zmiaikou's conjecture holds.*

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Proof.

Pair with $-\lambda_1^{\otimes 2} = \left[\frac{dx_1 \wedge dy_1}{y_1^2}\right]$:

$$2\chi(T_{d,\varepsilon=3}) + 3\chi(W_{d^2,\varepsilon=3}) + \chi(P_{d^2,\varepsilon=3}) = (1-d)\int_{X_{d^2}} dvol$$

□

Thank you very much for your attention!