Kudla's Modularity Conjecture and Formal Fourier Jacobi Series

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(joint work with Martin Raum)

Starting with the celebrated paper of Hirzebruch and Zagier [5] on intersection numbers of Hirzebruch-Zagier curves on Hilbert modular surfaces, the interplay of the geometry of special cycles on certain Shimura varieties and coefficients of modular forms has been a subject of active research with various applications.

Gross, Kohnen and Zagier proved in connection with their work on height pairings of Heegner points that the generating series of certain Heegner divisors on modular curves $X_0(N)$ is a (vector valued) modular form of weight $3/2$ with values in the first Chow group of $X_0(N)$, see [4]. A far-reaching generalization of this result for Shimura varieties associated with orthogonal groups was conjectured by Kudla in [7]. Here we briefly recall the background and report on recent results on the problem [10], [2], [9], [3].

Let (V, Q) be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$, and write $(\cdot, \cdot)$ for the bilinear form corresponding to Q. The hermitian symmetric space associated with the special orthogonal group $\mathrm{SO}(V)$ of V can be realized as

$$D = \{z \in V \otimes_{\mathbb{Q}} \mathbb{C} :\ (z, z) = 0 \text{ and } (z, \bar{z}) < 0\}/\mathbb{C}^\times.$$

This domain has two connected components. We fix one of them and denote it by D^+. Let $L \subset V$ be an even lattice. For simplicity we assume throughout this exposition that L is unimodular. This simplifies several technical aspects. For the general case we refer to [3]. Let $\Gamma \subset \mathrm{SO}(L)$ be a subgroup of finite index which takes D^+ to itself. The quotient

$$X_\Gamma = \Gamma \backslash D^+$$

has a structure as a quasi-projective algebraic variety of dimension n. It has a canonical model defined over a cyclotomic extension of $\mathbb{Q}$. For instance, if $n = 1$, then $\mathrm{SO}(V) \cong PB^\times$ for a quaternion algebra B over $\mathbb{Q}$ which is split at the archimedian place, D^+ is isomorphic to the upper complex half plane $\mathbb{H}$, and X_Γ is a (connected) Shimura curve.

There is a vast supply of algebraic cycles on X_Γ arising from embedded quadratic spaces $V' \subset V$ of smaller dimension. Let $1 \leq g \leq n$. For any $\lambda = (\lambda_1, \dots, \lambda_g) \in L^g$ with positive semi-definite inner product matrix $Q(\lambda) = \frac{1}{2}((\lambda_i, \lambda_j))_{i,j}$ there is a special cycle

$$Y_\lambda = \{z \in D^+ :\ (z, \lambda_1) = \cdots = (z, \lambda_g) = 0\}$$

on D^+, whose codimension is equal to the rank of $Q(\lambda)$. Its image in X_Γ defines an algebraic cycle, which we also denote by Y_λ. If $T \in \mathrm{Sym}_g(\mathbb{Q})$ is positive semi-definite of rank $r(T)$, we consider the special cycle on X_Γ of codimension $r(T)$ given by

$$Y(T) = \sum_{\substack{\lambda \in L^g/\Gamma \\ Q(\lambda) = T}} Y_\lambda,$$

see [6], [7]. We obtain a class in the Chow group $\mathrm{CH}^g(X_\Gamma)$ of codimension g cycles by taking the intersection pairing

$$Z(T) = Y(T) \cdot (\mathcal{L}^\vee)^{g-r(T)}$$

with a power of the dual of the tautological bundle $\mathcal{L}$ on X_Γ. Since the cycles Y_λ depend only on the orthogonal complement of the span of the vectors $\lambda_1, \dots, \lambda_g$, the cycles $Z(T)$ satisfy the symmetry condition

$$Z(T) = Z(u^t T u) \tag{1}$$

for all $u \in \mathrm{GL}_g(\mathbb{Z})$.

The following conjecture [7, Section 3, Problem 3] describes all rational relations among these cycles in an elegant way by means of a generating series on the Siegel upper half plane $\mathbb{H}_g$ of genus g. For $\tau \in \mathbb{H}_g$ we put $q^T = e(\mathrm{tr}(T\tau)) = \exp(2\pi i \,\mathrm{tr}(T\tau))$. The space of Siegel modular forms of weight k for the symplectic group $\mathrm{Sp}_g(\mathbb{Z})$ of genus g is denoted by $\mathrm{M}_k^{(g)}$.

Conjecture 1 (Kudla)**.** *The formal generating series*

$$A_g(\tau) = \sum_{\substack{T \in \mathrm{Sym}_g(\mathbb{Q}) \\ T \geq 0}} Z(T) \cdot q^T$$

is a Siegel modular form of weight $1 + n/2$ *for* $\mathrm{Sp}_g(\mathbb{Z})$ *with values in* $\mathrm{CH}^g(X_\Gamma)_\mathbb{C}$. *That is, for any linear functional* $h : \mathrm{CH}^g(X_\Gamma)_\mathbb{C} \to \mathbb{C}$, *the series*

$$h(A_g)(\tau) = \sum_{\substack{T \in \mathrm{Sym}_g(\mathbb{Q}) \\ T \geq 0}} h(Z(T)) \cdot q^T$$

is a Siegel modular form in $\mathrm{M}_{1+n/2}^{(g)}$.

The analogous statement for the cohomology classes in $H^{2g}(X_\Gamma)$ of the $Z(T)$ was proved by Kudla and Millson in a series of paper, see e.g. [8]. However, the cycle class map $\mathrm{CH}^g(X_\Gamma) \to H^{2g}(X_\Gamma)$ can have a large kernel. For instance, if $n = g = 1$ then its kernel consists of the subgroup of divisor classes of degree 0, which is arithmetically of great significance.

For modular curves $X_0(N)$, the above conjecture is true essentially by the Theorem of Gross-Kohnen-Zagier. For general n and codimension $g = 1$ (and arbitrary lattices and level) it was proved by Borcherds [1].

In the case of general n and general codimension g, Zhang proved the following partial modularity result. Write the variable $\tau \in \mathbb{H}_g$ and $T \in \mathrm{Sym}_g(\mathbb{Q})$ as block matrices

$$\tau = \begin{pmatrix} \tau_1 & z \\ z^t & \tau_2 \end{pmatrix}, \qquad T = \begin{pmatrix} n & r/2 \\ r^t/2 & m \end{pmatrix},$$

where $\tau_1 \in \mathbb{H}_1$, $\tau_2 \in \mathbb{H}_{g-1}$ and $z \in \mathbb{C}^{1\times(g-1)}$, and analogously $n \in \mathbb{Q}_{\geq 0}$, $m \in \mathrm{Sym}_{g-1}(\mathbb{Q})$ and $r \in \mathbb{Q}^{1\times(g-1)}$. Then we have $q^T = e(n\tau_1 + rz^t + \mathrm{tr}(m\tau_2))$. For fixed $m \in \mathrm{Sym}_{g-1}(\mathbb{Q})$ we consider the partial generating series

$$\phi_m(\tau_1, z) = \sum_{\substack{n \in \mathbb{Q}_{\geq 0} \\ r \in \mathbb{Q}^{1\times(g-1)}}} Z\begin{pmatrix} n & r/2 \\ r^t/2 & m \end{pmatrix} \cdot e(n\tau_1 + rz^t),$$

which can be viewed as the m-th formal Fourier-Jacobi coefficient of A_g. The truth of Kudla's conjecture would imply that $\phi_m(\tau_1, z)$ is a Jacobi form of index m. Zhang established this statement without assuming Kudla's conjecture. We write $J_{k,m}$ for the space of Jacobi forms of weight k and index m for the Jacobi group $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2\times(g-1)}$.

Theorem 2 (Zhang)**.** *The generating series $\phi_m(\tau_1, z)$ is a Jacobi form in $J_{1+n/2,m}$ with values in $\mathrm{CH}^g(X_\Gamma)_\mathbb{C}$, that is, an element of $J_{1+n/2,m} \otimes_\mathbb{C} \mathrm{CH}^g(X_\Gamma)_\mathbb{C}$.*

The proof of this result is based on the fact that $\phi_m(\tau_1, z)$ can be interpreted as a sum of push forwards of divisor generating series on embedded smaller orthogonal Shimura varieties of codimension $g-1$. The modularity of divisor generating series is known by Borcherds' result.

If we knew that the generating series A_g converged, then Zhang's theorem together with (1) would imply Kudla's conjecture, since the symplectic group is generated by translations, the discrete Levi factor $\mathrm{GL}_g(\mathbb{Z})$, and the embedded Jacobi group $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2\times(g-1)}$. However, there seems to be now direct way to obtain any such convergence result.

In [3] we prove a general modularity result for formal Fourier-Jacobi series, which implies the desired convergence statement. We now describe this.

A formal Fourier-Jacobi series of genus g (and weight k and cogenus $g-1$) is a formal series

$$f(\tau) = \sum_{\substack{m \in \mathrm{Sym}_{g-1}(\mathbb{Q}) \\ m \geq 0}} \phi_m(\tau_1, z) q_2^m,$$

with coefficients $\phi_m \in J_{k,m}$. Here $q_2^m = e(\mathrm{tr}\, m\tau_2)$ and $\tau_1 \in \mathbb{H}$, $z \in \mathbb{C}^{1\times(g-1)}$. We denote the Fourier coefficient of index (n, r) of ϕ_m by $c(\phi_m, n, r)$, and define the formal Fourier coefficient of f of index $T = \left(\begin{smallmatrix} n & r/2 \\ r^t/2 & m \end{smallmatrix}\right)$ by

$$c(f, T) = c(\phi_m, n, r).$$

The formal Fourier-Jacobi series f is called *symmetric*, if

$$c(f, T) = \det(u)^k \cdot c(f, u^t T u)$$

for all $u \in \mathrm{GL}_g(\mathbb{Z})$.

Theorem 3 (see [3])**.** *Every symmetric formal Fourier-Jacobi series of genus g and weight k converges, that is, it is the Fourier-Jacobi expansion of a Siegel modular form in $\mathrm{M}_k^{(g)}$.*

Corollary 4. *Conjecture 1 is true.*

Proof of the corollary. The result of Zhang shows that the generating series A_g is a formal Fourier-Jacobi series of weight $1 + n/2$ and genus g and cogenus $g - 1$. It is symmetric because of (1). Hence the claim follows from Theorem 3. $\square$

Corollary 5. *The subgroup of* $\mathrm{CH}^g(X_\Gamma)$ *generated by the classes* $Z(T)$ *for* $T \in \mathrm{Sym}_g(\mathbb{Q})$ *positive semi-definite has rank* $\leq \dim(\mathrm{M}^{(g)}_{1+n/2}$.

Note that it is not known in general whether the rank of $\mathrm{CH}^{(g)}(X_\Gamma)$ is finite.

Finally, we briefly comment on the idea of the proof of Theorem 3, referring to [3] for details. The space of Siegel modular forms $\mathrm{M}^{(g)}_k$ is a subspace of the space $\mathrm{FM}^{(g)}_k$ of symmetric formal Fourier-Jacobi series of weight k and genus g. In easy special cases (certain cases in genus 2) one can show that the dimensions of the two spaces agree and thereby prove the theorem. However, in general this seems hopeless. Instead, we use the symmetry condition and slope bounds for Siegel modular forms to compare the dimension asymptotics for $k \to \infty$. While $\dim(\mathrm{M}^{(g)}_k)$ grows like a positive constant times $k^{\frac{g(g+1)}{2}}$, we can establish the bound

$$\dim(\mathrm{FM}^{(g)}_k) \ll k^{\frac{g(g+1)}{2}}.$$

This implies that any $f \in \mathrm{FM}^{(g)}_k$ satisfies a non-trivial algebraic relation over the graded ring of Siegel modular forms. Now, viewing f as an element of the completion of the local ring $\hat{\mathcal{O}}_a$ at boundary points a of a regular toroidal compactification of X_Γ, one can deduce that f converges in a neighborhood of the boundary. Again using the algebraic relation, it can be shown that f has a holomorphic continuation to the whole domain $\mathbb{H}_g$.

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