The Macdonald identities and Jacobi forms of lattice index

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September 17, 2019

Modular Forms
on Higher Rank Groups
September 17–20, 2019
TU Darmstadt

Macdonald identities

Macdonald identity associated to an irreducible root system R

$$\prod_{n\geq 1}(1-X^n)^{\dim R}\prod_{r\in R}(1-X^n e^r)$$

$$=\sum_{x\in M} X^{(|x+w|^2-|w|^2)/4h}\frac{\sum_{g\in W(R)}\mathrm{sn}(g)\,e^{g(x+w)}}{\sum_{g\in W(R)}\mathrm{sn}(g)\,e^{gw}}$$

Notations

- w: half sum of positive roots
- $W(R)$: Weyl group
- $\mathrm{sn}(g)$: determinant of g
- M: lattice generated by $hr^\vee$ ($r^\vee = 2r/|r|^2$)
- h: $\frac{1}{2}(|\alpha+w|^2-|w|^2)$ (α highest root)

Specializations of the Macdonald identities

- Setting $e^r = 1$ and $X = e^{2\pi i\tau}$:

$$\eta(\tau)^{\dim R + |R|} = \sum_{x \in w+M} d(x) X^{|x|^2/2h} \quad (d(x) = \prod_{r \in R^+} \frac{(x,r)}{(w,r)}).$$

- Clearing the denominator and using the Weyl denominator formula

$$\sum_{g \in W(R)} \mathrm{sn}(g)\, e^{-gw} = \prod_{r \in R^+} (e^{r/2} - e^{-r/2}),$$

the Macdonald id. can be viewed as generalization of Weyl's formulas.

- Taking $R = A_1$ yields the *Jacobi triple product identity*:

$$\prod_{n \geq 1} (1 - X^n)(1 - X^n e^{\alpha})(1 - X^n e^{-\alpha}) = \sum_{m \in \mathbb{Z}} (-1)^m X^{m(m-1)/2} e^{-m\alpha}.$$

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Jacobi's theta function I

Jacobi triple product identity

$$\vartheta(\tau, z) = \sum_{r\in\mathbb{Z}} \left(\frac{-4}{r}\right) q^{\frac{r^2}{8}} \zeta^{\frac{r}{2}} = \eta(\tau) q^{\frac{1}{12}} (\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n\geq 1} (1 - q^n\zeta)(1 - q^n\zeta^{-1})$$

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1} (1 - q^n) \qquad (q = \mathrm{e}(\tau) = \mathrm{e}^{2\pi i\tau},\ \zeta = \mathrm{e}(z),\ \tau \in \mathbb{H},\ z \in \mathbb{C}).$$

Note

- $$\vartheta(\tau, z) = \eta(\tau)^3 \sigma(\tau, z),$$
  where $\sigma(\tau, z)$ is the Weierstrass σ-function of $\mathbb{Z}\tau + \mathbb{Z}$.
- This explains in essence the Jacobi triple product identity.

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Jacobi's theta function II

Question

Can the cited Macdonald identities also interpreted and explained along these lines?

Anticipated answer

Observation

- *The formal power series occurring in the Macdonald identities can indeed be interpreted as distinguished functions:* Jacobi forms of lattice index*.*
- *They are of* singular weight*: Such functions are rare, they are essentially invariants of* Weil representations*.*
- *The theory of Jacobi forms of lattice index will provide a* simple *explanation for the product expansion (without even referring to zeros or divisors).*

Plan

Plan of the talk

- Describe "precisely" *Jacobi forms of lattice index*.
- Describe how product expansions just "happen" by *pulling back Jacobi forms* of the simplest type.
- Describe the Jacobi forms proof of the Macdonald identities.
- Discuss some natural questions emerging from that proof.
- Give some puzzling applications to explicit number theory (elliptic curves over the rationals).

The group $\mathrm{Mp}(2,\mathbb{Z})$

$$\mathrm{Mp}(2,\mathbb{Z}) := \left\{(A,w) : A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z}),\ w(\tau) = \pm\sqrt{c\tau + d}\right\},$$

where $(A,w)\cdot(B,v) = (AB, w(B\tau)v(\tau))$, defines an extension of $\mathrm{SL}(2,\mathbb{Z})$:

$$1 \to \{(1,\pm 1)\} \xrightarrow{\subseteq} \mathrm{Mp}(2,\mathbb{Z}) \xrightarrow{(A,w)\mapsto A} \mathrm{SL}(2,\mathbb{Z}) \to 1.$$

The application

$$\varepsilon(A,w) := \eta(A\tau)/w(\tau)\eta(\tau)$$

defines a linear character $\varepsilon : \mathrm{Mp}(2,\mathbb{Z}) \to \mu_{24}$.

Theorem

The group of linear characters of $\mathrm{Mp}(2,\mathbb{Z})$ *is cyclic of order 24, generated by* ε.

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Lattices

Basic notions

- (Integral positive) Lattice $\underline{L} = (L, \beta)$:
  - Finite free $\mathbb{Z}$-module L,
  - symmetric, positive definite $\mathbb{Z}$-bilinear map $\beta : L \times L \to \mathbb{Z}$.
- Set
$$\beta(x) := \frac{1}{2}\beta(x, x).$$
- $\underline{L}$ is *even* if $\beta(x)$ integral for all x in L, *odd* otherwise.
- $x \mapsto \beta(x) \bmod \mathbb{Z}$ defines a linear character $L \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.
- The shadow of $\underline{L}$ is
$$L^\bullet = \{s \in \mathbb{Q} \otimes L : \beta(s, x) \equiv \beta(x) \bmod \mathbb{Z} \text{ for all } x \text{ in } L\}.$$
($L^\bullet$ equals the dual $L^\sharp$ of L if $\underline{L}$ is even.)
- $L_{\text{ev.}}$ denotes the kernel of $x \mapsto \beta(x) \bmod \mathbb{Z}$: maximal even sublattice.

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Jacobi forms of lattice index I

Definition

$J_{k,\underline{L}}\left(\varepsilon^h\right)$ (k integral or haf-integral, h integer mod 24): space of holomorphic functions $\phi(\tau, z)$ ($\tau \in \mathbb{H}$, $z \in \mathbb{C} \otimes_{\mathbb{Z}} L$) such that:

1. For all (A, w) in $\mathrm{Mp}(2, \mathbb{Z})$,

$$\{\phi|_{k,\underline{L}}(A, s)\}(\tau, z) := \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) e\left(\frac{-c\beta(z)}{c\tau + d}\right) w(\tau)^{-2k} = \varepsilon(A, w)^h\, \phi(\tau, z).$$

2. For $x, y \in L$,

$$\phi(\tau, z + \tau x + y)\ \mathrm{e}\left(\tau\beta(x) + \beta(z, x)\right) = e\left(\beta(x + y)\right)\, \phi(\tau, z).$$

3. ϕ holomorphic at infinity.

Jacobi forms of lattice index II

Fourier expansion

ϕ is called *holomorphic at infinity* if its Fourier expansion is of the form

$$\phi = \sum_{\substack{n \in \frac{h}{24}+\mathbb{Z},\ r \in L^\bullet \\ n \geq \beta(r)}} c_\phi(n,r)\, q^n \,\mathrm{e}\left(\beta(r,z)\right).$$

Theta expansion

For $\lambda : L^\bullet / L_{\text{ev.}} \to \mathbb{C}$ such that $\lambda(y+x) = \lambda(y) e\left(\beta(x)\right)$ for all $y \in L^\bullet$, $x \in L$, set

$$\vartheta_{\underline{L},\lambda} := \sum_{r \in L^\bullet} \lambda(r) q^{\beta(r)}\, e\left(\beta(r,z)\right)$$

and let

$$\Theta(\underline{L}) := \operatorname{span}_{\mathbb{C}} \left\{ \vartheta_{\underline{L},\lambda} : \lambda \text{ as above} \right\}.$$

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Theta expansion (cont.)

$\mathrm{Mp}(2,\mathbb{Z})$ acts on $\Theta(\underline{L})$ via $(g,\theta) \mapsto \theta|_{\frac{n}{2},\underline{L}}g^{-1}$. One has

$$J_{k,\underline{L}}\left(\varepsilon^h\right) \cong \left(M_{k-\frac{n}{2}}(\Gamma(\ell)) \otimes \Theta(\underline{L}) \otimes \mathbb{C}\left(\varepsilon^h\right)\right)^{\mathrm{Mp}(2,\mathbb{Z})},$$

where $n = \operatorname{rank} \underline{L}$, $\ell =$level of $\underline{L}$.

Consequences

- $J_{k,\underline{L}}\left(\varepsilon^h\right)$ is finite dimensional.
- $J_{k,\underline{L}}\left(\varepsilon^h\right) = 0$ for $k < \frac{n}{2}$.
- $J_{\frac{n}{2},\underline{L}}\left(\varepsilon^h\right) = \left(\Theta(\underline{L}) \otimes \mathbb{C}\left(\varepsilon^h\right)\right)^{\mathrm{Mp}(2,\mathbb{Z})}$.
  ($\frac{n}{2}$: singular weight; $\Theta(\underline{L})$ Weil representations associated to the discriminant module of $\underline{L}$ (for even $\underline{L}$ ...).

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Examples

- Let $\underline{\mathbb{Z}}(2m) := (\mathbb{Z}, (x,y) \mapsto 2mxy)$. Then $J_{k,\underline{\mathbb{Z}}(2m)}(1)$ equals "classical" $J_{k,m}$.
- $\vartheta(\tau, z)$ defines an element of $J_{\frac{1}{2},\frac{1}{2}}\left(\varepsilon^3\right)$.
- Set
$$\underline{\mathbb{Z}}^N = \left(\mathbb{Z}^N, (x,y) \mapsto x \cdot y\right).$$
The function
$$\vartheta_{\underline{\mathbb{Z}}^N}(\tau, z) := \vartheta(\tau, z_1)\vartheta(\tau, z_2)\cdots\vartheta(\tau, z_N)$$
$(z = (z_1, z_2, \ldots, z_N) \in \mathbb{C} \otimes \mathbb{Z}^N)$ defines an element of $J_{\frac{N}{2},\underline{\mathbb{Z}}^N}\left(\varepsilon^{3N}\right)$.

A useful construction

A simple effective construction method

Any isometric embedding $\alpha : \underline{L} \to \underline{L}'$ yields a map (*pullback*)

$$\alpha^* : J_{k,\underline{L}'}\left(\varepsilon^h\right) \to J_{k,\underline{L}}\left(\varepsilon^h\right), \quad \{\alpha^*\phi\}(\tau, z) = \phi(\tau, \alpha(z)).$$

In particular, any

$$\alpha = (\alpha_1, \ldots, \alpha_N) : \underline{L} \to \underline{\mathbb{Z}}^N$$

yields the Jacobi form

$$\{\alpha^*\vartheta_{\underline{\mathbb{Z}}^N}\}(\tau, z) := \vartheta(\tau, \alpha_1(z)) \cdots \vartheta(\tau, \alpha_N(z))$$

in $J_{\frac{N}{2},\underline{L}}\left(\varepsilon^{3N}\right)$.

Some remarks on embeddings

Remarks

- The number of N and embeddings $\alpha : \underline{L} \to \underline{Z}^N$ is finite — if one does not permit embeddings with some α_j identically zero.
  (Indeed, if z_j are coordinate functions with respect to a $\mathbb{Z}$-basis of L, the $\alpha_j(z)$ become linear forms in z_j with integral coefficients whose squares add up to the quadratic form $\beta(z)$.)
- Q: Which lattices permit embeddings into $\underline{\mathbb{Z}}^N$ (Conway-Sloane: *integrable lattices*)?
- A (Conway-Sloane): All of rank ≤ 5. The lattice E_6 is not integrable.
- How does one compute effectively all embeddings of a given lattice?

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A related non-trivial problem I

- Q: Given $\alpha : \underline{L} \to \underline{Z}^N$, by what power η^l can we divide such that $\alpha^* \vartheta_{\underline{\mathbb{Z}}^N} / \eta^l$ is still holomorphic at infinity, and defines therefore a Jacobi form in $J_{\frac{N-l}{2},\underline{L}}\left(\varepsilon^{3N-l}\right)$? (The best we can have is $l = N - n$.)
- A: $\alpha^* \vartheta_{\underline{Z}^N} / \eta^l$ is holomorphic at infinity if and only if

$$B_\alpha(x) := \sum_{j=1}^{N} B\left(\alpha_j(x)\right) \geq \frac{l}{24}$$

for all x in $\mathbb{R} \otimes L$. Here

$$B(x) = \frac{1}{2}\left(y - \frac{1}{2}\right)^2, \quad \text{where} \quad y \equiv x \bmod \mathbb{Z},\ 0 \leq y \leq 1.$$

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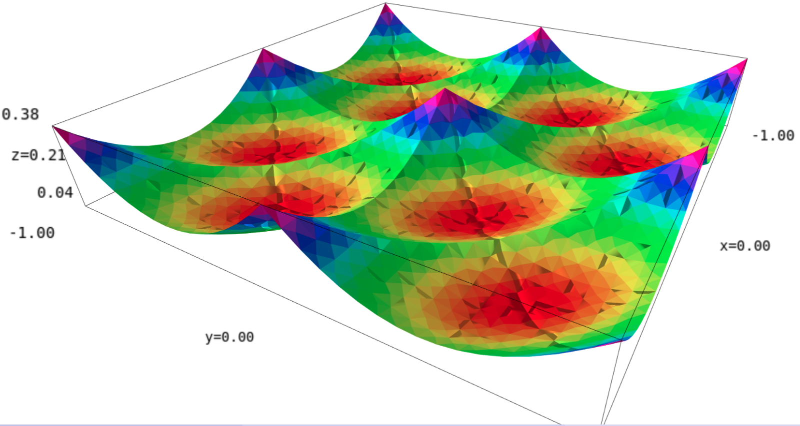
A related non-trivial problem II

Question

Is there an effective way to compute the minimum of the continuous, piecewise differentiable, piecewise quadratic, and periodic function B_α?



Eutactic stars I

Remark

For every embedding $\alpha = (\alpha_1, \dots, \alpha_N)$ there exist s_j in $L^\sharp$ such that $\alpha_j(x) = \beta(s_j, x)$.

Proposition

For any family $s = \{s_j\}_{1 \le j \le N}$ of elements in $L^\sharp$ the following are equivalent:

1. $\beta(x, x) = \sum_{j=1}^{N} \beta(s_j, x)^2$ *for all x in L.*
2. $\beta(x, y) = \sum_{j=1}^{N} \beta(s_j, x)\beta(s_j, y)$ *for all x in L.*
3. $x = \sum_{j=1}^{N} \beta(s_j, x)\, s_j$ *for all x in L.*

Definition

A family $s = \{s_j\}_j$ of nonzero elements s_j in L satisfying prop. (1)–(3) is called *eutactic star on L*.

Eutactic stars II

Note

For any eutactic star s on L,

- the map $x \mapsto (\beta(s_1, x), \ldots, \beta(s_N, x))$ defines an embedding $\alpha : \underline{L} \to \underline{Z}^N$.
- The function

$$\vartheta_s(\tau, z) := \vartheta\left(\tau, \beta_1(s_1, z)\right) \cdots \vartheta\left(\tau, \beta_1(s_N, z)\right)$$

  defines an element of $J_{\frac{N}{2}, \underline{L}}\left(\varepsilon^{3N}\right)$ (since it equals $\alpha^* \vartheta_{\underline{\mathbb{Z}}^N}$).
- We look for *extremal* eutactic stars, i.e., for those such that

$$\min_{x \in \mathbb{R} \otimes L} \sum_{j=1}^{N} B\left(\beta(s_j, x)\right) \geq \frac{N - n}{24}.$$

Extremal eutactic stars I

- Let G be a subgroup of g in $O(\underline{L})$ leaving the s_j invariant up to sign and including multiplicities (i.e., of g for which there exists a permutation σ and signs ϵ_j such that $gs_j = \epsilon_j s_{\sigma(j)}$). For g in G set

$$\mathrm{sn}(g) = \prod_{j=1}^{N} \epsilon_j.$$

- This is independent of the choice of σ, and defines a linear character of G.
- Note: $O(\underline{L})$ (and, in particular, G) acts on $\Theta(\underline{L})$ and on every $J_{k,\underline{L}}\left(\varepsilon^h\right)$ via pullback.
- This action intertwines with the action of $\mathrm{Mp}(2,\mathbb{Z})$.
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Extremal eutactic stars II

Definition

s is called *G-extremal* if

$$\Theta(\underline{L})^{G,\mathrm{sn}} := \{\theta \in \Theta(\underline{L}) : g^*\theta = \mathrm{sn}(g)\theta\}$$

is one-dimensional.

Remark

1. $\Theta(\underline{L})^{G,\mathrm{sn}}$ is at least one-dimensional. (It contains $\vartheta_{\underline{L},\lambda_D}$, where $\lambda_D(r)$ is the coefficient of $q^{\beta(r)-D}\, e\,(\beta(z,r))$ in ϑ_s.)
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Extremal eutactic stars II

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The Macdonald type identities generating theorem

Theorem

Let $\underline{L} = (L, \beta)$ be an integral lattice of rank n, let s be a G-extremal eutactic star on $\underline{L}$. Then there is a constant γ and a vector w in $L^\bullet$ such that

$$\eta^{n-N} \prod_{j=1}^{N} \vartheta\big(\tau, \beta(s_j, z)\big) = \gamma \sum_{x \in w + L_{ev.}} q^{\beta(x)} \sum_{g \in G} \mathrm{sn}(g)\, e\big(\beta(gx, z)\big).$$

In particular, the product on the left defines an element of the space of Jacobi forms $J_{n/2,\underline{L}}(\varepsilon^{n+2N})$.

Sketch of proof

Proof.

- Right hand side is obviously in $\Theta(\underline{L})^{G,\mathrm{sn}}$; by assumption there is a w such that the RHS is nonzero: call this function ϕ_s.
- For any k and h, one has

$$J_{k,\underline{L}}\left(\varepsilon^h\right)^{G,\mathrm{sn}} = \text{Modular form on } \mathrm{SL}(2,\mathbb{Z}) \times \phi_s.$$

  Modular form on $\mathrm{SL}(2,\mathbb{Z})$ means polynomial in η, E_4 and E_6. (This follows from carefully looking at the theta decomposition.)
- In particular $\vartheta_s = f \times \phi_s$ for some modular form f.
- Since ϑ_s does not vanish identically in z for any τ, it follows f is an η-power.

The theorem is now obvious. $\square$

A simple example

Example

$S : s_1 = 1$ defines an eutactic star on $\underline{\mathbb{Z}}$. Let $G = O(\underline{Z}) = \{\pm 1\}$. We have $Z_{\text{ev.}} = 2\mathbb{Z}$ and $\mathbb{Z}^\bullet = \frac{1}{2}\mathbb{Z}$, and $L^\bullet/L_{\text{ev.}} = \{\frac{1}{2} + 2\mathbb{Z}, -\frac{1}{2} + 2\mathbb{Z}\}$. So there is only one orbit anyway and the assumptions of the theorem are trivially fulfilled.
The resulting identity is the triple product identity.

Root systems as eutactic stars

- Let R be an irreducible root system with ambient Euclidean space $(R, (\cdot,\cdot))$, R^+ a system of positive roots.
- Well-known identity: $h\,(x,x) = \sum_{r\in R^+}(r,x)^2$ $(x \in E)$, where $h = \frac{1}{n}\sum_{r\in R^+}(r,r)$.
- Set $\underline{R} = (W, (\cdot,\cdot)/h)$ where $W = \{x \in E : (x,r)/h \in \mathbb{Z} \text{ for all } r \in R\}$.
- Thus: R^+ is an eutactic star on $\underline{R}$.
- Let G be the Weyl group of R.

Theorem

The eutactic star R^+ on $\underline{R}$ is extremal with respect to the Weyl group G of R.

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Proof that R^+ is Weyl group extremal

It suffices to show:

- The stabilizer of $w + W_{\text{ev.}}$ (under G) is contained in the kernel of sn.
- Any class not in the Weyl group orbit of $w + W_{\text{ev.}}$ is stabilized by a reflection.

Lemma

Let v be any element in $W^\bullet$ which has minimal length among all elements in $v + W_{ev.}$.

1. *One has $(\alpha, v) \leq h$.*
2. *If $(\alpha, v) = h$ then $v \equiv g_\alpha(v) \bmod W_{ev.}$, where g_α is the reflection through the hyperplane perpendicular to α.*
3. *If $(\alpha, v) < h$ and $v \in C$, then $v = w$.*

(Here α is the highest root and C the Weyl chamber associated to R^+.)

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The Macdonald identities as identity for Jacobi forms

Theorem

Let R be an irreducible root system with a choice of positive roots R^+, and let w be the half sum of the positive roots of R. Then, in the notations of the preceding paragraphs, we have

$$\vartheta_R(\tau, z) := \eta(\tau)^{n-N} \prod_{r \in R^+} \vartheta\big(\tau, (r, z)/h\big) = \sum_{x \in w + W_{ev.}} q^{(x,x)/2h} \sum_{g \in G} \mathrm{sn}(g)\, e\big((gx, z)/h\big)$$

for all τ is the upper half plane and all z in $\mathbb{C} \otimes W$. The function ϑ_R defines in particular a holomorphic Jacobi form in $J_{n/2,\underline{R}}(\varepsilon^{n+2N})$. (Here $n = \dim E$, $N = |R^+|$.)

Construction of classical Jacobi forms I

- We have seen ϑ_R in $J_{\frac{n}{2},\underline{R}}\left(\varepsilon^{n+2N}\right)$ ($n = \dim R$, $N = |R^+|$.)
- Especially interesting: $n = 4$, $N \equiv -2 \bmod 12$, since then $\vartheta_R \in J_{2,\underline{R}}(1)$, and
- every a in $\underline{R}_{\text{ev.}}$ yields embedding $\alpha_a : \underline{\mathbb{Z}}\left((a, a)/h\right) \to \underline{R}$ (via $x \mapsto xa$), and
- thus $\alpha_a^* \vartheta_{\underline{R}} = \alpha_a^* \left(\alpha_{R^+}^* \vartheta_{\underline{\mathbb{Z}}^{|R^+|}}\right) / \eta^{n-N}$ is an element of $J_{2,(a,a)/2h}$, which is a space of classical Jacobi forms closely connected to the arithmetic theory of modular forms (and e.g., elliptic curves over the rationals).

Construction of classical Jacobi forms II

Table: The four infinite families $\vartheta_R(\tau,(a,b,c,d)z)$ of theta blocks of weight 2 and trivial character associated to root systems (We write ϑ_n for $\vartheta(\tau,nz)$.)

R	$\vartheta_R(\tau,(a,b,c,d)z)$
A_4	$\eta^{-6}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_{a+b+c}\,\vartheta_{a+b+c+d}\,\vartheta_b\,\vartheta_{b+c}\,\vartheta_{b+c+d}\,\vartheta_c\,\vartheta_{c+d}\,\vartheta_d$
$G_2 \oplus B_2$	$\eta^{-6}\,\vartheta_a\,\vartheta_{3a+b}\,\vartheta_{3a+2b}\,\vartheta_{2a+b}\,\vartheta_{a+b}\,\vartheta_b\,\vartheta_c\,\vartheta_{c+d}\,\vartheta_{c+2d}\,\vartheta_d$
$A_1 \oplus B_3$	$\eta^{-6}\,\vartheta_a\,\vartheta_b\,\vartheta_{b+c}\,\vartheta_{b+2c+2d}\,\vartheta_{b+c+d}\,\vartheta_{b+c+2d}\,\vartheta_c\,\vartheta_{c+d}\,\vartheta_{c+2d}\,\vartheta_d$
$A_1 \oplus C_3$	$\eta^{-6}\,\vartheta_a\,\vartheta_b\,\vartheta_{2b+2c+d}\,\vartheta_{b+c}\,\vartheta_{b+2c+d}\,\vartheta_{b+c+d}\,\vartheta_c\,\vartheta_{2c+d}\,\vartheta_{c+d}\,\vartheta_d$

Table: The 42 one-dimensional $J_{2,m}$ generated by a newform, given as pullback, and associated elliptic curve.

m	CL	Curve	Theta block
37	37a1	$y^2 + y = x^3 - x$	$\vartheta_1^3\vartheta_2^3\vartheta_3^2\vartheta_4\vartheta_5$
43	43a1	$y^2 + y = x^3 + x^2$	$\vartheta_1^3\vartheta_2^2\vartheta_3^2\vartheta_4^2\vartheta_5$
53	53a1	$y^2 + xy + y = x^3 - x^2$	$\vartheta_1^3\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5\vartheta_6$
57	57a1	$y^2 + y = x^3 - x^2 - 2x + 2$	$\vartheta_1^2\vartheta_2^2\vartheta_3^3\vartheta_4\vartheta_5\vartheta_6$
58	58a1	$y^2 + xy = x^3 - x^2 - x + 1$	$\vartheta_1^2\vartheta_2^3\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6$
61	61a1	$y^2 + xy = x^3 - 2x + 1$	$\vartheta_1^2\vartheta_2^3\vartheta_3^2\vartheta_4\vartheta_5\vartheta_7$
65	65a1	$y^2 + xy = x^3 - x$	$\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5^2\vartheta_6$
77	77a1	$y^2 + y = x^3 + 2x$	$\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5\vartheta_6\vartheta_7$
79	79a1	$y^2 + xy + y = x^3 + x^2 - 2x$	$\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5^2\vartheta_8$
82	82a1	$y^2 + xy + y = x^3 - 2x$	$\vartheta_1\vartheta_2^3\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6\vartheta_7$
83	83a1	$y^2 + xy + y = x^3 + x^2 + x$	$\vartheta_1^2\vartheta_2\vartheta_3^2\vartheta_4^2\vartheta_5\vartheta_6\vartheta_7$
88	88a1	$y^2 = x^3 - 4x + 4$	$\vartheta_1^2\vartheta_2^2\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6\vartheta_8$
89	89a1	$y^2 + xy + y = x^3 + x^2 - x$	$\vartheta_1^3\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_6^2\vartheta_7$

Open questions

- Are there extremal eutactic stars other than the root systems?
- One could think of a systematic search: 40,000 lattices in the Sloane-Nebe database . . .
- How does one compute effectively all embeddings of a given lattice into $\underline{\mathbb{Z}}^N$?
  - For a given integral quadratic form Q in n variables find all decompositions as sum of squares of integral linear forms.
  - Naive search: computationally expensive. (Expon. in the determinant.)
  - So far I only could do this for lattices with determinant ≤ 100.
- To be sure not to miss any Macdonald type identity one would need to compute $\min_{x\in\mathbb{R}\otimes L}\sum_{j=1}^{N} B\left(\alpha_j(x)\right)$ for any given embedding $\alpha : \underline{L} \to \underline{\mathbb{Z}}^N$. But . . .
- How many modular forms of weight 2 and trivial character can one obtain via pullback based on

$$\underline{\mathbb{Z}}(2m) \to \underline{R} \to \mathbb{Z}^{10} \quad (R = A_4, G_2 \oplus B_2, A_1 \oplus B_3, A_1 \oplus C_3)?$$

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Thank you!