Finding all paramodular Borcherds products and applications

with emphasis on rigorous computations

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Modular Forms on Higher Rank Groups
Technical University of Darmstadt, September 19 2019

The take away of this talk

Joint work with Jerry Shurman and Cris Poor

1. We developed an algorithm to find every (degree 2) paramodular cusp form of a fixed weight k and level N that is a Borcherds product. (Prior to spanning the space $S_k(K(N))$ of paramodular cusp forms.)
2. We implemented this algorithm and ran some examples and applications.

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Acknowledgement: Thank you to Valery Gritsenko for teaching us how to make Borcherds products!

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- Sine function (Euler)

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$$\eta(\tau) = q^{1/24} \prod_{n \in \mathbb{N}} (1 - q^n)$$

[$\epsilon\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in e\left(\frac{1}{24}\right)$ is chosen to make $\epsilon\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sqrt{c\tau + d}$ a factor of automorphy on $\mathrm{SL}(2,\mathbb{Z}) \times \mathcal{H}_1$.]

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- Are there useful infinite products in many variables?

Infinite Products II

- Yes!
- Odd Jacobi Theta function $\vartheta \in J^{\mathrm{cusp}}_{1/2,1/2}(\epsilon^3 v_H)$

$$\vartheta(\tau, z) = q^{1/8}\left(\zeta^{1/2} - \zeta^{-1/2}\right)\prod_{n\in\mathbb{N}}(1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1})$$

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  $\zeta = 1$ or $\exists n \in \mathbb{N} : q^n\zeta^{\pm 1} = 1 \iff z \in \mathbb{Z}\tau + \mathbb{Z}$
- Richard Borcherds has a theory of infinite products that are automorphic forms for $O(2, n)$.

Definition of Siegel Modular Forms

- Siegel Upper Half Space: $\mathcal{H}_n = \{Z \in M^{\mathrm{sym}}_{n\times n}(\mathbb{C}) : \operatorname{Im} Z > 0\}$.
- Symplectic group: $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ acts on $Z \in \mathcal{H}_n$ by $\sigma \cdot Z = (AZ+B)(CZ+D)^{-1}$.
- $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{R})$ such that $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ has finite index in Γ and $\mathrm{Sp}_n(\mathbb{Z})$
- Slash action: For $f : \mathcal{H}_n \to \mathbb{C}$ and $\sigma \in \mathrm{Sp}_n(\mathbb{R})$, $(f|_k\sigma)(Z) = \det(CZ+D)^{-k} f(\sigma \cdot Z)$.
- Siegel Modular Forms: $M_k(\Gamma)$ is the $\mathbb{C}$-vector space of holomorphic $f : \mathcal{H}_n \to \mathbb{C}$ that are "bounded at the cusps" and that satisfy $f|_k\sigma = f$ for all $\sigma \in \Gamma$.
- Cusp Forms: $S_k(\Gamma) = \{f \in M_k(\Gamma)$ that "vanish at the cusps"$\}$

Definition of paramodular form

- A *paramodular form* is a Siegel modular form for a paramodular group. In degree 2, the paramodular group of level N, is

$$\Gamma = K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \qquad * \in \mathbb{Z},$$

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- $K(N)$ is the stabilizer in $\mathrm{Sp}_2(\mathbb{Q})$ of $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$.
- ${}^T K(N) \backslash \mathcal{H}_2$ is a moduli space for complex abelian surfaces with polarization type $(1, N)$. (T is "transpose" here.)
- The paramodular Fricke involution splits paramodular forms into plus and minus spaces.

$$S_k\left(K(N)\right) = S_k\left(K(N)\right)^+ \oplus S_k\left(K(N)\right)^-$$

Fourier-Jacobi expansion (FJE)

$$\text{FJE: } f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{m \in \mathbb{Z}:\, m \geq 0} \phi_m(\tau, z) e(Nm\omega)$$

The Fourier-Jacobi expansion of a paramodular form is fixed *term-by-term* by the following subgroup of the paramodular group $K(N)$:

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

- $P_{2,1}(\mathbb{Z})/\{\pm I\} \cong \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathrm{Heisenberg}(\mathbb{Z})$

Thus the coefficients ϕ_m are automorphic forms in their own right and easier to compute than Siegel modular forms. This is one motivation for the introduction of Jacobi forms.

Definition of Jacobi Forms: Automorphicity

Level one

- Assume $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is holomorphic.

$$\begin{aligned} E_m\phi : \mathcal{H}_2 &\to \mathbb{C} \\ \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) &\mapsto \phi(\tau, z)e(m\omega) \end{aligned}$$

- Assume that $E_m\phi$ transforms by $\chi \det(CZ + D)^k$ for

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

Definition of Jacobi Forms: Support

- Jacobi forms are tagged with additional adjectives to reflect the support $\mathrm{supp}(\phi) = \{(n, r) \in \mathbb{Q}^2 : c(n, r; \phi) \neq 0\}$ of the Fourier expansion

$$\phi(\tau, z) = \sum_{n,r \in \mathbb{Q}} c(n, r; \phi) q^n \zeta^r, \qquad q = e(\tau), \zeta = e(z).$$

- $\phi \in J_{k,m}^{\mathrm{cusp}}$: automorphic and $c(n, r; \phi) \neq 0 \implies 4mn - r^2 > 0$

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- $\phi \in J_{k,m}$: automorphic and $c(n, r; \phi) \neq 0 \implies 4mn - r^2 \geq 0$
- $\phi \in J^{\mathrm{weak}}_{k,m}$: automorphic and $c(n, r; \phi) \neq 0 \implies n \geq 0$

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- $\phi \in J_{k,m}^{\mathrm{weak}}$: automorphic and $c(n, r; \phi) \neq 0 \implies n \geq 0$
- $\phi \in J_{k,m}^{\mathrm{wh}}$: automorphic and $c(n, r; \phi) \neq 0 \implies n \gg -\infty$ ("wh" stands for *weakly holomorphic*)

Borchherds Product Summary

Theorem (Borcherds, Gritsenko, Nikulin)

Given $\psi \in J^{\mathrm{wh}}_{0,N}(\mathbb{Z})$*, a weakly holomorphic weight zero, index N Jacobi form with integral coefficients*

$$\psi(\tau, z) = \sum_{n,r\in\mathbb{Z}:\ n\geq -N_o} c\,(n,r)\, q^n \zeta^r$$

there is a weight $k' \in \mathbb{Z}$*, a character* χ*, and a* meromorphic *paramodular form* $\mathrm{Borch}(\psi) \in M^{\mathrm{mero}}_{k'}(K(N))(\chi)$

$$\mathrm{Borch}(\psi)(Z) = q^A \zeta^B \xi^C \prod_{n,m,r\in\mathbb{Z}} \left(1 - q^n \zeta^r \xi^{Nm}\right)^{c(nm,r)}$$

converging in a nbhd of infinity and defined by analytic continuation.

Borchers Product Theorem Details

- A, B, C are explicitly calculated from the q^0 term of ψ.

$$A = 1/24 \sum_{r \in \mathbb{Z}} c(0, r)$$

$$B = 1/2 \sum_{r \in \mathbb{Z}_{\geq 1}} rc(0, r)$$

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- Whether Borch(ψ) is a Fricke plus or minus form is calculated from the principal part of ψ.

Theta Blocks: a great way to make Jacobi forms

due to Gritsenko, Skoruppa, and Zagier

- Dedekind Eta function $\eta \in J^{\mathrm{cusp}}_{1/2,0}(\epsilon)$
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- Shorthand notation: $0^e = \eta(\tau)^e$ and $d^e = (\vartheta(\tau, dz)/\eta(\tau))^e$
- Theta block $0^{2k} d_1^{e_1} d_2^{e_2} \cdots d_\ell^{e_\ell} \in J^{\mathrm{mero}}_{k,m}(\epsilon^{2k+2\sum_i e_i})$

  where $2m = e_1 d_1^2 + e_2 d_2^2 + \cdots + e_\ell d_\ell^2$ and $d_i \in \mathbb{N}$, $e_i \in \mathbb{Z}$.

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- Theorem (G.-S.-Z.) on when a theta block is in $J^{\text{cusp}}_{k,m}$
- For example, $\phi_1, \phi_2, \phi_3 \in J^{\text{cusp}}_{2,277}$

$$\phi_1 = 0^4 1^2 2^2 3^2 4^1 5^1 14^1 17^1$$

$$\phi_2 = 0^4 1^1 3^1 4^2 5^1 6^1 8^1 9^2 15^1$$

$$\phi_3 = 0^4 1^1 2^1 3^1 4^2 5^1 7^1 8^1 9^1 17^1$$

An example of a Borcherds product in $S_2(K(277))$

- A paramodular cusp form of weight 2 and paramodular level 277

$$\mathrm{Borch}(\psi)(Z) = q\,\zeta^{28}\xi^{277} \prod_{(m,n,r)\geq 0} (1 - q^n \zeta^r \xi^{277m})^{c(nm,r)}$$

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- The $c(n,r)$ are given by a certain weakly holomorphic Jacobi cusp form $\psi \in J^{\mathrm{wh}}_{0,277}(\mathbb{Z})$

$$\psi = -\frac{\phi_1|V_2}{\phi_1} - \frac{\phi_2|V_2}{\phi_2} + \frac{\phi_3|V_2}{\phi_3}, \qquad \psi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n,r)q^n\zeta^r.$$

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- We use the index raising operator $V_2 : J_{k,m} \to J_{k,2m}$ from Eichler-Zagier.

An example of a Borcherds product in $S_2(K(277))$ II

- The expansion of the weakly holomorphic Jacobi form

$$\begin{aligned}\psi(\tau, z) &= \sum_{n,r\in\mathbb{Z}} c(n,r)q^n\zeta^r, \\ &= 4 + 2\zeta + \zeta^2 + 2\zeta^3 + \zeta^4 + \zeta^5 + \zeta^6 - \zeta^7 + \zeta^9 + \zeta^{14} + \zeta^{15} \\ &+ q(\zeta^{34} + \zeta^{35} + \zeta^{36} + \cdots) + q^2(\zeta^{48} + \cdots) + q^3(-\zeta^{58} + \cdots) \\ &+ q^5\zeta^{75} - q^9\zeta^{100} + q^{11}\zeta^{111} + q^{12}\zeta^{116} + q^{14}\zeta^{125} + q^{20}\zeta^{149} \\ &+ q^{31}\zeta^{186} + q^{35}\zeta^{197} + q^{36}\zeta^{200} + \cdots\end{aligned}$$

(only singular terms shown)

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- Borch(ψ) vanishes on 23 Humbert surfaces, 19 with order one, and the rest with orders $2, 4, 5, 10$. For example,

$$\mathrm{Hum}_{277}(29, 197) = \{Z = \left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right) : 35\tau + 197z + 277\omega = 0\}$$

(Note here $29 = 197^2 - 4 \cdot 35 \cdot 277$.)

The algorithm: Summary

Goal: Make $\mathrm{Borch}(\psi) = \phi\xi^{cN} + \phi_2\xi^{(c+1)N} + \cdots \in S_k(K(N))$.

(Details of these steps to follow.)

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Step 3 Find (initial expansions of) all $\phi_2 \in J_{k,(c+1)N}^{\mathrm{cusp}}$ with ϕ_2/ϕ holomorphic

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Step 3 Find (initial expansions of) all $\phi_2 \in J_{k,(c+1)N}^{\mathrm{cusp}}$ with ϕ_2/ϕ holomorphic

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(Details of these steps to follow.)

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Step 5 If $\psi \in J_{0,N}^{\mathrm{wh}}(\mathbb{Z})$ really exists then $\Delta^j\psi \in J_{12j,N}^{\mathrm{cusp}}$ for $j > -\operatorname{ord}(\psi_{\mathrm{maybe}})$.
Span $J_{12j,N}^{\mathrm{cusp}}$ and see if any of the initial expansions match $\Delta^j\psi_{\mathrm{maybe}}$.

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Step 6 Create $\text{Borch}(\psi) \in M_k(K(N))$ and check whether it is a cusp form.

(Details of these steps to follow.)

The algorithm: Step 1

Goal: Make $\mathrm{Borch}(\psi) = \phi\xi^{cN} + \phi_2\xi^{(c+1)N} + \cdots \in S_k(K(N))$.

- To determine the possible range of c above, bound the number of initial Fourier-Jacobi coefficients that can vanish in $S_k(K(N))$.

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- If the space $S_k(K(N))$ is not known, we can use the following theorem:

Theorem (Breeding, Poor, Yuen)

Let $f \in S_k(K(N))^{\pm}$. Let m be the minimum of some formulas too gnarly to typeset here. If the first m Fourier-Jacobi coefficients of f vanish, then f must vanish.

See: Breeding, Poor, Yuen, *Computations of Spaces of Paramodular Forms of General Level*, JKMS (2016).

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We now loop through the possible c.

The algorithm: Step 2

Goal: Make $\mathrm{Borch}(\psi) = \phi\xi^{cN} + \phi_2\xi^{(c+1)N} + \cdots \in S_k(K(N))$.

- The leading Fourier Jacobi coefficient must be a theta block.

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- The leading Fourier Jacobi coefficient must be a theta block.
- For each possible c from step 1, we find all theta blocks $\phi \in J_{k,cN}^{\text{cusp}}$. One can find all finite sequences $d_1, e_1, \ldots, d_\ell, e_\ell$ of integers such that

$$2cN = e_1 d_1^2 + \cdots + e_\ell d_\ell^2$$

  that satisfy certain additional conditions when some e_i are negative, and satisfy the conditions for the theta block to be a Jacobi cusp form.
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The algorithm: Step 3

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- From $\operatorname{Borch}(\psi) = \phi \exp(-\operatorname{Grit}(\psi))$, we have $\psi = -\phi_2/\phi$. We now find candiate $\phi_2 \in J^{\mathrm{cusp}}_{k,(c+1)N}$ such that ϕ_2/ϕ is holomorphic.

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Theorem (Poor, Shurman, Yuen)

Let m be some formula in the d's and e's that make up the theta block ϕ. If the first m terms of ϕ_2 are divisible by ϕ, then ϕ_2 is divisible by ϕ.

- But in practice we do not use this theorem, and do Step 5 instead.

The algorithm: Step 4

Goal: Make $\mathrm{Borch}(\psi) = \phi\xi^{cN} + \phi_2\xi^{(c+1)N} + \cdots \in S_k(K(N))$.

- We now have a space of initial expansions of candidate ψ.

The algorithm: Step 4

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Now loop over all candidates ψ_{maybe}.

The algorithm: Step 5

Goal: Make $\mathrm{Borch}(\psi) = \phi\xi^{cN} + \phi_2\xi^{(c+1)N} + \cdots \in S_k(K(N))$.

- We need to verify that ψ_{maybe} is really the truncation of some $\psi \in J_{0,N}^{\mathrm{wh}}(\mathbb{Z})$. This is because it is usually intractable to expand to the required guaranteeing length in Step 3.

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- Span $J_{12j,N}^{\mathrm{cusp}}$ and see if any of the initial expansions match $\Delta^j\psi_{\mathrm{maybe}}$. If yes, then we have found a ψ such that $\mathrm{Borch}(\psi) \in M_k(K(N))$. If no, then ψ_{maybe} can be discarded.

The algorithm: Step 6

Goal: Make $\mathrm{Borch}(\psi) = \phi\xi^{cN} + \phi_2\xi^{(c+1)N} + \cdots \in S_k(K(N))$.

- We now have a set of ψ where $\mathrm{Borch}(\psi) \in M_k(K(N))$. This set includes all the possible ψ for which $\mathrm{Borch}(\psi) \in S_k(K(N))$.

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Theorem (Poor, Shurman, Yuen)

An $f \in M_k(K(N))$ is a cusp form if and only if some explicit set of finitely many Fourier coefficients of the form

$$a\left(n\delta \begin{bmatrix} 1 & -m \\ -m & m^2 \end{bmatrix}; f\right)$$

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End of algorithm.

Example: All Borcherds products in $S_9(K(16))$

- (Screenshot from our website www.siegelmodularforms.org)

$\varphi \in J_{9,c\cdot 16}^{cusp}$ $q^{c+t} \,\|\, \varphi$	**t = 0**	**t = 1**
c = 1	$1^{-5}2^{7}3^{1}$: S S S M M M $1^{-1}2^{2}3^{1}4^{1}$: S S S S	$1^{11}2^{3}3^{1}$: S
c = 2	$1^{2}2^{11}3^{2}$: S $1^{7}2^{3}3^{5}$: S M $1^{6}2^{6}3^{2}4^{1}$: S S $1^{10}2^{1}3^{2}4^{2}$: S $1^{9}2^{3}3^{2}5^{1}$: S M $1^{11}2^{2}3^{1}6^{1}$: ∅	$1^{18}2^{7}3^{2}$: ∅
c = 3	$1^{13}2^{10}3^{3}4^{1}$: S $1^{14}2^{7}3^{6}$: ∅ $1^{17}2^{5}3^{3}4^{2}$: ∅ $1^{16}2^{7}3^{3}5^{1}$: ∅ $1^{18}2^{6}3^{2}6^{1}$: ∅	

Note $2cN = \sum_i e_i d_i^2$ and $2k + 2\sum_i e_i = 24(c+t)$, where $k = 9$, $N = 16$ here. We omitted the 0^{18} part of each theta block.

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- Borcherds products were used to span and determine the space $S_9(K(16))$. This was extra challenging because 16 is highly non-square-free.
- An eigenform in $S_9(K(16))$ was found to have eigenvalues that proves it generates an automorphic representation with a supercuspidal 2-component. As far as we know, this is the first example of generating a supercuspidal component. Level 16 is the smallest level where this can happen, and weight 9 is the lowest weight where such an eigenform exists.
  (See: Cris Poor, Ralf Schmidt, David Yuen: *Paramodular forms of level 16 and supercuspidal representations*, to appear in Moscow Journal of Combinatorics and Number Theory.)

Application: Modularity of Abelian Surfaces

Degree 1: All elliptic curves $E/\mathbb{Q}$ are modular

Modularity Theorem

(Wiles; Wiles & Taylor; Breuil, Conrad, Diamond & Taylor)

Let $N \in \mathbb{N}$. To each elliptic curve $E/\mathbb{Q}$ with conductor N there exists a normalized Hecke eigenform $f \in S_2(\Gamma_0(N))^{\mathrm{new}}$ with rational eigenvalues such that

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- Eichler (1954) proved the first examples $L(X_0(11), s, \mathrm{Hasse}) = L(\eta(\tau)^2\eta(11\tau)^2, s, \mathrm{Hecke})$.

"All abelian surfaces $A/\mathbb{Q}$ with a minimal endomorphism group over $\mathbb{Q}$ are paramodular."

Paramodular Conjecture (Brumer and Kramer 2009)

Let $N \in \mathbb{N}$. To each abelian surface $A/\mathbb{Q}$ with conductor N and endomorphism ring $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$, there exists a Hecke eigenform $f \in S_2(K(N))^{\mathrm{new}}$ that has rational eigenvalues and is not a Gritsenko lift from $J_{2,N}^{\mathrm{cusp}}$ such that

$$L(A, s, \mathrm{Hasse\text{-}Weil}) = L(f, s, \mathrm{spin}).$$

Application: Modularity of Abelian Surfaces

Degree 2: Paramodular conjecture

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Note: the original converse of this conjecture has been amended.

Modularity proven for $N = 277, 353, 587^-$

(*On the paramodularity of typical abelian surfaces,* Brumer, Pacetti, Poor, Tornarìa, Voight, Yuen - on arXiv and soon to be published in the Journal of Algebra and Number Theory)

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- Hecke eigenvalue computer calculations for paramodular eigenforms written as rational functions of Gritsenko lifts and Borcherds products.
- Galois representations associated to automorphic representations whose archimedean component is a holomorphic limit of discrete series.

The role of Borcherds products in this application

- Gritsenko lifts always land in the symmetric (plus for even weights and minus for odd weights) Fricke space. Borcherds products can land in either the symmetric or antisymmetric Fricke space.

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- We expect Borcherds products to play a crucial role in future modularity proofs.

Weight $k = 2$

- Paramodular Conjecture of Brumer and Kramer: the modularity of abelian surfaces defined over $\mathbb{Q}$ with minimal endomorphisms is shown by weight two nonlift paramodular newforms with rational eigenvalues.

N	$\dim J_{2,N}^{\mathrm{cusp}}$	$\dim S_2(K(N))$	various comments
249	5	6	BP+Grit; Jac
277	10	11	modular! Q/L; Jac
295	6	7	BP+Grit; Jac
349	11	12	BP+Grit; Jac
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- Poor, Shurman, Yuen have some (partly rigorous, partly heuristic) tables up to $N \leq 1000$

Heuristic tables: $k = 2$ paramodular newforms: $N \le 800$.

$$+\text{new nonlift} = \dim\left(\left(S_2(K(N))^{\text{new}}\right)^+ \,/\, \text{Grit}\left(J_{2,N}^{\text{cusp}}\right)\right)$$

$$-\text{new} = \dim\left(S_2(K(N))^{\text{new}}\right)^- .$$

The "=" means "proven."

N	+new nl	−new	various comments
249	$= 1$		BP+Grit; Jac
277	$= 1$		modular! Q/L; Jac
295	$= 1$		BP+Grit; Jac
349	$= 1$		BP+Grit; Jac
353	$= 1$		modular! BP+Grit; Jac
388	1		Jac

N	$+$new nl	$-$new	various comments
389	$=1$		BP$+$Grit; Jac
394	1		Jac
427	1		Jac
461	$=1$		Tr(BP)$+$Grit; Jac
464	1		Jac
472	1		Jac
511	2		quad pair, $\sqrt{5}$; 4-dim $A/\mathbb{Q}$?
523	$=1$		BP$+$Grit; Jac
550	1		Prym (A. Sutherland)

N	$+$new nl	$-$new	various comments
555	1		Jac
561	1		Prym
574	1		Jac
587	$=1$	$=1$ *modular!*	Tr(BP)+Grit and BP-; Jacs
597	1		Jac
603	1		Jac
604	1		Jac
623	1		Jac
633	1		Jac

N	$+$new nl	$-$new	various comments
637	2		quad pair, $\sqrt{2}$; 4-dim $A/\mathbb{Q}$?
644	1		Jac
645	2		quad pair, $\sqrt{2}$; 4-dim $A/\mathbb{Q}$?
657	≥ 1		modular!* WR: $E_{(9\zeta_6-8)}/\mathbb{Q}(\sqrt{-3})$
665	1		Prym
688	1		Jac
691	1		Jac
702	1		Prym (A. Sutherland)

$$\zeta_6 = \exp(2\pi i \tfrac{1}{6})$$

* via the lift of Berger, Dembélé, Pacetti, Seguin from a Bianchi modular form to a paramodular form

N	$+$new nl	$-$new	various comments
704	1		Jac
708	1		Jac
709	1		Jac
713	1	≥ 1	BP-; Jacs
731	$= 1$		Berger and Klosin: modular!* Poor, Shurman, Yuen Jac
737	1		Prym
741	1		Jac
743		1	Jac

* Tobias Berger and Krzysztof Klosin: *Deformations of Saito-Kurokawa type and the Paramodular Conjecture*, with appendix by Poor, Shurman, Yuen, on arXiv and to appear in American Journal of Math.

N	+new nl	−new	various comments
745	1		Jac
760	1		Prym (A. Sutherland)
762	1		Jac
763	1		Jac
768	1		Jac
775	≥ 1		modular!* WR: $E_{(5\phi-2)}/\mathbb{Q}(\sqrt{5})$
797	1		Jac

$$\phi = (1+\sqrt{5})/2$$

* via the lift of Johnson-Leung and Roberts from a Hilbert modular form to a paramodular form

The Paramodular conjecture 2.0 (2018)

An abelian fourfold $B/\mathbb{Q}$ has *quaternionic multiplication* (QM) if $\mathrm{End}_{\mathbb{Q}}(B)$ is an order in a non-split quaternion algebra over $\mathbb{Q}$. A cuspidal, nonlift Siegel paramodular newform $f \in S_2(K(N))$ with rational Hecke eigenvalues will be called a *suitable* paramodular form of level N.

Paramodular Conjecture (Brumer–Kramer)

Let $N \in \mathbb{N}$. Let $\mathcal{A}_N$ be the set of isogeny classes of abelian surfaces $A/\mathbb{Q}$ of conductor N with $\mathrm{End}_{\mathbb{Q}} A = \mathbb{Z}$. Let $\mathcal{B}_N$ be the set of isogeny classes of QM abelian fourfolds $B/\mathbb{Q}$ of conductor N^2. Let $\mathcal{P}_N$ be the set of suitable paramodular forms of level N, up to nonzero scaling. There is a bijection $\mathcal{A}_N \cup \mathcal{B}_N \leftrightarrow \mathcal{P}_N$ such that

$$L(C, s, \mathrm{H\text{-}W}) = \begin{cases} L(f, s, \mathrm{spin}), & \text{if } C \in \mathcal{A}_N, \\ L(f, s, \mathrm{spin})^2, & \text{if } C \in \mathcal{B}_N. \end{cases}$$

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Brumer and Kramer: QM implies $N = M^2 s$ with $s \mid \gcd(30, M)$.

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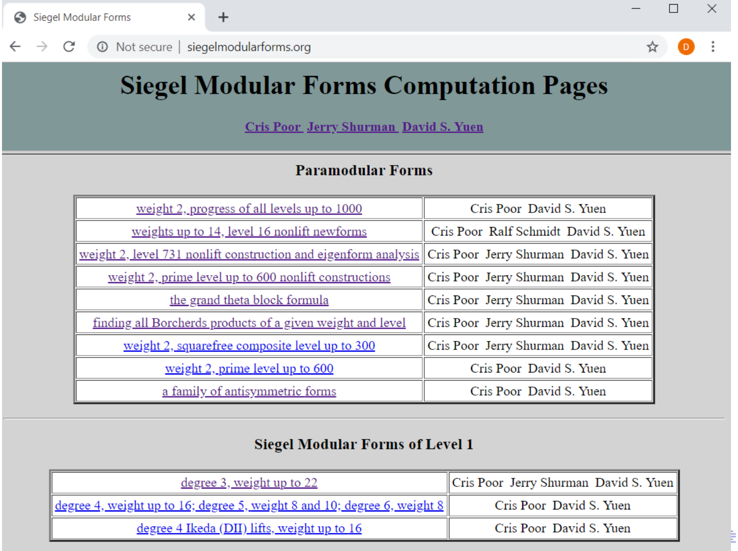

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Siegel Modular Forms Computation Pages

Cris Poor Jerry Shurman David S. Yuen

Paramodular Forms

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weights up to 14, level 16 nonlift newforms	Cris Poor Ralf Schmidt David S. Yuen
weight 2, level 731 nonlift construction and eigenform analysis	Cris Poor Jerry Shurman David S. Yuen
weight 2, prime level up to 600 nonlift constructions	Cris Poor Jerry Shurman David S. Yuen
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weight 2, prime level up to 600	Cris Poor David S. Yuen
a family of antisymmetric forms	Cris Poor David S. Yuen

Siegel Modular Forms of Level 1

degree 3, weight up to 22	Cris Poor Jerry Shurman David S. Yuen
degree 4, weight up to 16; degree 5, weight 8 and 10; degree 6, weight 8	Cris Poor David S. Yuen
degree 4 Ikeda (DII) lifts, weight up to 16	Cris Poor David S. Yuen

Thank you!