Critical L-values and congruences for Siegel modular forms, II

Abhishek Saha

Queen Mary University of London

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Main result from Ameya's talk

Theorem (Pitale-S-Schmidt)

Let π be a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ such that π_∞ is isomorphic to the holomorphic discrete series representation with highest weight (k_1, k_2) such that $k_1 \geq k_2 \geq 3, k_1 \equiv k_2 \pmod 2$. Let χ be any Dirichlet character satisfying $\chi(-1) = (-1)^{k_1}$. Let r be any integer satisfying $1 \leq r \leq k_2 - 2, r \equiv k_2 \pmod 2$; if $\chi^2 = 1$, assume that $r > 1$. Let F be a vector-valued Siegel modular form of weight $\det^{k_2} \mathrm{sym}^{k_1 - k_2}$ associated to π.

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$$\frac{L(r, \pi \boxtimes \chi, \mathrm{std})}{(2\pi i)^{2k+3r} G(\chi)^3 \langle F, F \rangle} \in \overline{\mathbb{Q}},$$

and the quantity on the left is $\mathrm{Aut}(\mathbb{C})$ equivariant.

Natural question: What primes divide the numerator/denominator of this quantity? Are they of arithmetic significance?

Some further motivation

Special values for the Riemann zeta function:
If m is an even integer than

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Recall also:

- $\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty}(1 - e^{2\pi inz})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz}$.
- $\sigma_r(n) = \sum_{d|n} d^r$, the Fourier coefficients of E_{r+1}

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A miraculous congruence (observed by Ramanujan in 1916)

The Fourier coefficients of Δ and E_{12} are congruent modulo 691. In other words, $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

Congruences

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- In the previous example, the primes dividing $\zeta(12)/\pi^{12}$ turn out to congruence primes for E_{12} (The Eisenstein series may be viewed as lifts of the "trivial" character).
- What can be say about more general modular forms and L-functions?

Some notation for this talk:
Let p be a prime. Then we let $v_p : \overline{\mathbb{Q}_p} \to \mathbb{Q} \cup \infty$ denote the p-adic valuation normalized by $v_p(p) = 1$. By fixing an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, this allows us to define v_p on $\overline{\mathbb{Q}}$.
Given $\alpha, \beta \in \overline{\mathbb{Q}}$, we say

$$p|\alpha \quad \text{if} \quad v_p(\alpha) > 0,$$

$$\alpha \text{ is } p\text{-integral} \quad \text{if} \quad v_p(\alpha) \geq 0,$$

$$\alpha \equiv \beta \pmod{p} \quad \text{if} \quad v_p(\alpha - \beta) > 0.$$

Warning: $v_p(\alpha) > 0$ merely means that *some* prime $\mathfrak{p}$ lying *above* p in *some* number field divides $\alpha - \beta$.

Congruence between modular forms

Notations (contd.)
Let F, G be two modular forms (classical, Siegel,..)
Suppose that they are both Hecke eigenforms away from some integer N, with Hecke eigenvalues $\lambda_F(T)$, $\lambda_G(T)$ for all $T \in \mathcal{H}(N)$.

We say that

$$F \equiv G \pmod{p},$$

if

$$v_p\left(\lambda_F(T) - \lambda_G(T)\right) > 0 \text{ for all } T \in \mathcal{H}(N).$$

Warning: We do not insist that the system of Hecke eigenvalues are congruent at all places, but at *almost all* places.

A result of Katsurada for classical newforms

Theorem (Katsurada, 2005)

Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, and χ be an even Dirichlet character. Let $p > (2k-1)!$ be a prime number such that, for some odd integer r, $1 \leq r \leq k-1$,

$$p \mid \frac{\pi^{k+2r}\langle f, f\rangle}{L(r, \mathrm{sym}^2 f \times \chi)}.$$

Then there exists a normalized Hecke eigenform $g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, such that $f \neq g$ and

$$f \equiv g \pmod{p}.$$

Remark: Katsurada also had a version when the level is a prime.

An older result of Hida

Theorem (Hida, 1981)

Let $f \in S_k(\Gamma_0(N))$ be a normalized newform, where $N \geq 3$. Let r_0 be the dimension of the space generated by all Galois conjugates of f. Let

$$c(f) := \frac{L(1, \mathrm{sym}^2 f)}{u(f)\pi^{r_0(k+1)}}$$

where $u(f)$ is a certain *period*. Then $c(f)^2 \in \mathbb{Q}$. Let $p > k-2$ be a prime number such that $p \nmid N$ and

$$p \mid c(f).$$

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Denominators vs numerators

We can reframe Katurada's result using the well-known formula

$$\langle f, f \rangle = \frac{L(1, \mathrm{sym}^2 f) N (k-1)!}{2^{2k+1} \pi^{k+1}}$$

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Hida

If $p \geq k-1$, $p \nmid N$, p divides $\frac{L(1, \mathrm{sym}^2 f)}{u(f) \pi^{k+1}}$ then p is a congruence prime for f.

Katsurada, reframed

If $p > (2k-1)!$, $p \nmid N$, p divides $\frac{\pi^{2r-1} L(1, \mathrm{sym}^2 f)}{L(r, \mathrm{sym}^2 f \times \chi)}$ for suitable r and χ then p is a congruence prime for f.

What can one say for Siegel modular forms?

Previous results for Siegel modular forms of degree 2:

- Congruence primes for general Siegel modular forms of degree 2: Katsurada (2008)
- Between Saito-Kurokawa lifts and non-lifts: Brown (2007, 2011), Katsurada (2008), Brown-Agarwal (2014).
- Between Yoshida lifts and non-lifts: Böcherer–Dummigan–Schulze-Pillot (2012), Agarwal–Klosin (2013).

All the above results have restrictions on the level (usually full level or squarefree level), and on the archimedean type (usually scalar valued forms). Some of these results has applications to the Bloch-Kato conjecture for various L-functions.

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Today's talk: Some ideas to generalise the above works and get a general congruence theorem for arbitrary levels and archimedean types and then reframe it using the GGP conjecture.

Key idea

- Recall from previous talk: The "pullback" formula:

$$\langle E^{\chi}_{k,N}(- , Z_2, s, \bar{F}_0\rangle \approx L(2s + k - n, \pi \boxtimes \chi, \varrho_{2n+1}),$$

  where $\approx$ means we omit some explicit quantities. (Relevant range: $s = -m_0$ where $0 \leq m_0 \leq \frac{k-n-1}{2}$ is an integer.)

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- Recall also, for $n = 2$,:

$$N_k(\Gamma_4(N))^\circ = \bigoplus_{\substack{\ell=2 \\ \ell\equiv k \bmod 2}}^{k} \bigoplus_{\substack{m=0 \\ m\equiv 0 \bmod 2}}^{k-\ell} D_+^{(k-\ell-m)/2}\, U^{m/2}\, S_{\ell,m}(\Gamma_4(N))$$

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- So, if we can prove that $E^{\chi}_{k,N}(Z_1, Z_2, -m_0)$ is cuspidal in each variable, we will be able to "expand" it in terms of various F_0.

Arithmetic of pullback of Eisenstein series in degree n

In fact, we do it for any degree n:

Theorem

Let $k \geq 2n+2$, $0 \leq m_0 \leq \frac{k}{2} - n - 1$. Let $p \nmid 2N$ be a prime such that $p \geq 2k$. Then the nearly holomorphic modular form on $\mathbb{H}_n \times \mathbb{H}_n$

$$E(Z_1, Z_2) := \pi^{n+n^2-(2n+1)k+(2n+2)m_0} \Lambda^N\left(\frac{k-2m_0}{2}\right) E^{\chi}_{k,N}(Z_1, Z_2, -m_0)$$

is cuspidal in each variable, and all of it's Fourier coefficients are p-integral.

Above,

$$\Lambda^N(s) := L^N(2s, \chi) \prod_{i=1}^{n} L^N(4s - 2i, \chi^2).$$

Key ideas behind proof

For **cuspidality**:

- Strategy follows Garrett (who dealt with the case $m_0 = 0$).

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- For general translates of E, use the *q-expansion principle*.
- For general m_0 use arithmetic of differential operators.

So we get:

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Now combine this with:

$$\langle E^{\chi}_{k,N}(\,-\,, Z_2, s, \bar{F}_0\rangle \approx L(2s+k-n, \pi \boxtimes \chi, \varrho_{2n+1}),$$

$$N_k(\Gamma_4(N))^\circ = \bigoplus_{\substack{\ell=2 \\ \ell \equiv k \bmod 2}}^{k} \; \bigoplus_{\substack{m=0 \\ m \equiv 0 \bmod 2}}^{k-\ell} D_+^{(k-\ell-m)/2}\, U^{m/2}\, S_{\ell,m}(\Gamma_4(N)).$$

The key formula in degree 2

- Let us denote by $\mathcal{B}_{\ell,m,N}$ an orthogonal basis of Hecke eigenforms for $S_{\ell,m}(\Gamma_4(N))$.
- Let $\mathcal{C}_{\ell,m,k,N} := \{D_+^{(k-\ell-m)/2}\, U^{m/2} F : F \in \mathcal{B}_{\ell,m,N}\} \subset N_k(\Gamma_4(N))^\circ$.

We can obtain an explicit differential operator $T_{\ell+m,\ell}$, such that

The explicit pullback formula

$$T_{\ell+m,\ell} E^{\chi}_{k,N}(Z_1, Z_2, -m_0) \approx$$
$$\times \sum_{F \in \mathcal{C}_{\ell,m,k,N}} \frac{L^N(k - 2m_0 - 2, \pi_F \boxtimes \chi, \varrho_5)}{\langle \bar{F}, \bar{F} \rangle} \bar{F}(Z_1) F(Z_2).$$

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- This is perfectly suited for proving congruences between Hecke eigenvalues of Siegel cusp forms of different types (lifts and non-lifts)

A linear algebra trick (Katsurada)

- Let $F_i \in S_k(\Gamma_4(N)), 1 \leq i \leq d$ be linearly independent Hecke eigenforms with Hecke eigenvalues $\lambda_i(T)$, $T \in \mathcal{H}(N)$, and Fourier coefficients $A_i(S)$, with $\lambda_i(T) \in \overline{\mathbb{Z}}$.
- Suppose $G \in S_k(\Gamma_4(N))$ with Fourier coefficients $A_G(S) \in \overline{\mathbb{Z}}$, such that
$$G(Z) = \sum_i c_i F_i(Z), \qquad c_i \in \overline{\mathbb{Q}}.$$

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Let p be a prime. Assume that

1. For all S, the Fourier coefficients of G satisfy $v_p(A_G(S)) \geq 0$.
2. There is a S_0 such that $v_p(A_1(S_0)) = 0$.
3. We have $v_p(c_1) < 0$.

A linear algebra exercise

There is a $i \neq 1$ such that

$$\lambda_1(T) \equiv \lambda_i(T) \pmod{p} \text{ for all } T \in \mathcal{H}(N).$$

Various lifts to $S_\rho(\Gamma)$

- Saito-Kurokawa (CAP) lifts F. For these, there exists a classical newform f such that

$$L_{st}(s, F) \approx L(s + \frac{1}{2}, f)L(s - \frac{1}{2}, f)\zeta(s).$$

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- Yoshida lifts. For these there exist two classical newforms f_1, f_2 such that

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- Endoscopic lifts from $\mathrm{GL}_2(K)$, where K is a quadratic field. These may be viewed as the non-split version of Yoshida lifts.

Arthur packets

- We say that two automorphic representations $\pi \cong \otimes\pi_p$ and $\pi' \cong \otimes\pi'_p$ of $\mathrm{GSp}_4(\mathbb{A})$ are nearly equivalent, if $\pi_p \cong \pi'_p$ for almost all primes p.
- It is known that the near-equivalence classes are precisely the global Arthur packets.
- Our condition $\ell \geq 3$ implies that elements of $\Pi_N(\ell + m, \ell)$ belong to Arthur packets of type **(G)** (general type), **(Y)** (Yoshida type) or **(P)** (Saito-Kurokawa type).
- The local Arthur packets for types **(G)**, **(Y)** and **(P)** have the property that unramified representations lie in 1-element packets. It follows that if $\pi, \pi' \in \Pi_N(\ell + m, \ell)$ are nearly equivalent, then in fact $\pi_p \cong \pi'_p$ for all $p \nmid N$.

Congruences between Arthur packets

Let ψ_1, ψ_2 be two Arthur packets. We say that

$$\psi_1 \equiv \psi_2 \pmod{p},$$

if there exists some N such that

$$v_p\left(\lambda_{\psi_1}(T) - \lambda_{\psi_2}(T)\right) > 0 \text{ for all } T \in \mathcal{H}(N),$$

i.e., the Hecke eigenvalues of ψ_1 and ψ_2 coincide at all but finitely many places.

- Recall that $\mathcal{B}_{\ell,m,N}$ is an orthogonal basis of Hecke eigenforms for $S_{\ell,m}(\Gamma_4(N))$.
- $\mathcal{C}_{\ell,m,k,N} := \{D_+^{(k-\ell-m)/2}\, U^{m/2} F : F \in \mathcal{B}_{\ell,m,N}\} \subset N_k(\Gamma_4(N))^\circ$.
- Now, we have

$$S_{\ell,m}(\Gamma(N)) = \bigoplus_{\psi} S_{\ell,m}(\Gamma(N))_\psi,$$

where ψ runs over all possible Arthur parameters.

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$$S_{\ell,m}(\Gamma(N)) = \bigoplus_{\psi} S_{\ell,m}(\Gamma(N))_\psi,$$
where ψ runs over all possible Arthur parameters.
- Define $\mathcal{B}_{\ell,m,N,\psi}$ to be the subset of $\mathcal{B}_{\ell,m,N}$ lying in $S_{\ell,m}(\Gamma(N))_\psi$, and $\mathcal{C}_{\ell,m,k,N,\psi} = \{\pi^{-m/2} U^{m/2} F : F \in \mathcal{B}_{\ell,m,N,\psi}\}$.

Given some matrix S,

$$C_{\psi,N}(S) = \sum_{F \in \mathcal{C}_{\ell,m,k,N,\psi}} \frac{|a(S,F)|^2}{\langle F, F\rangle}. \tag{1}$$

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We can now state a first version of our main theorem.

Theorem

Let $k_1 \geq k_2 \geq 6$ be integers of the same parity. Let ψ_1 be an Arthur parameter that intersects the archimedean type associated to the holomorphic discrete series (k_1, k_2).

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Let $k_1 \geq k_2 \geq 6$ be integers of the same parity. Let ψ_1 be an Arthur parameter that intersects the archimedean type associated to the holomorphic discrete series (k_1, k_2). Now, suppose there exists:

- *An integer $4 \leq r \leq k_2 - 2$ of the same parity as k_1*
- *A Dirichlet character χ of the same parity as k_1*
- *An integer N divisible by the primes ramified either in ψ_1 or χ*
- *A prime number $p > (3k_1)^4$ such that $p \nmid N$*

such that for some S

$$v_p\Big(\frac{L^N(r, \psi_1 \boxtimes \chi, \rho_5) C_{\psi_1,N}(S)}{\pi^{2k+3r}}\Big) < 0. \tag{2}$$

Then there exists an Arthur parameter ψ_2, distinct from ψ_1 such that

$$\psi_1 \equiv \psi_2 \pmod{p} \tag{3}$$

Theorem

Let $k_1 \geq k_2 \geq 6$ be integers of the same parity. Let ψ_1 be an Arthur parameter that intersects the archimedean type associated to the holomorphic discrete series (k_1, k_2). Now, suppose there exists:

- *An integer $4 \leq r \leq k_2 - 2$ of the same parity as k_1*
- *A Dirichlet character χ of the same parity as k_1*
- *An integer N divisible by the primes ramified either in ψ_1 or χ*
- *A prime number $p > (3k_1)^4$ such that $p \nmid N$*

such that for some S

$$v_p\left(\frac{L^N(r, \psi_1 \boxtimes \chi, \rho_5) C_{\psi_1,N}(S)}{\pi^{2k+3r}}\right) < 0. \quad (2)$$

Then there exists an Arthur parameter ψ_2, distinct from ψ_1 such that

$$\psi_1 \equiv \psi_2 \pmod{p} \quad (3)$$

Remark: Can replace S by $\overline{\mathbb{Z}}$-linear combinations.

Fourier coefficients and Bocherer's conjecture

The Gan-Gross-Prasad conjecture for $(\mathrm{SO}(5), \mathrm{SO}(2))$

Let π be an automorphic representation of $\mathrm{PGSp}(4)$ inside an Arthur packet for types **(G)** or **(Y)**, E a quadratic field, and Λ a character of $\mathbb{A}_E^\times$ such that $\Lambda|_{\mathbb{A}^\times} = 1$. Let ϕ be any automorphic form in the space of π, and let N be an integer outside which everything is unramified. Then

$$\frac{|B(\phi,\Lambda)|^2}{\langle \phi, \phi \rangle} = \frac{C_T}{2^\ell} \frac{\zeta(2)\zeta(4)L(\frac{1}{2}, \pi \otimes \Lambda)}{L(1,\pi,\mathrm{Ad})L(1,\chi_d)} \prod_{p|N \text{ or } p=\infty} \alpha_p(\phi_p, \Lambda_p).$$

where $\alpha_p(\phi_p, \Lambda_p)$ is an explicit local integral, equal to 1 almost everywhere.

This is now a theorem by work of Furusawa and Morimoto.

The local integral for an Arthur packet

Warning: Everything from here is *work in progress*.

Given a tempered local Arthur packet (L-packet) ψ_p, a local character Λ_p of $E_p^\times$, and an integer N, define

$$\alpha_{p,N}(\psi_p, \Lambda_p) = \sum_{\pi_p \in \psi_p} \sum_{\phi_p \in V_{\pi_p}(K(N))} \alpha_p(\phi_p, \Lambda_p),$$

where $\alpha_p(\phi_p, \Lambda_p)$ is the local integral occurring in the GGP formula.

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where $\alpha_p(\phi_p, \Lambda_p)$ is the local integral occurring in the GGP formula.

- This is a purely local quantity.
- It is a non-negative real number that is also *algebraic*.
- Given ψ_p and Λ_p, there exists some N such that $\alpha_{p,N}(\psi_p, \Lambda_p) \neq 0$ **if and only if** there exists some $\pi_p \in \psi_p$ has a Λ_p-type Bessel model.

Theorem (refined version)

Let $k_1 \geq k_2 \geq 6$ be integers of the same parity. Let ψ_1 be an Arthur parameter of Type $(\mathbf{G})$ that intersects the archimedean type associated to the holomorphic discrete series (k_1, k_2).

Theorem (refined version)

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$$v_p\Big(\frac{\pi^{2k+3r} L^N(1, \psi_1, \mathrm{Ad})}{L^N(r, \psi_1 \boxtimes \chi, \rho_5) L^N(\frac{1}{2}, \psi_1 \times \Lambda) \alpha_\infty(k_1, k_2) \prod_{p|N} \alpha_{p,N}(\psi_p, \Lambda_p)}\Big) > 0.$$

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Then there exists an Arthur parameter ψ_2, distinct from ψ_1 such that

$$\psi_1 \equiv \psi_2 \pmod{p} \tag{4}$$

Theorem (Yoshida lifts)

Let f and g be two distinct classical newforms of weights ℓ_1, ℓ_2 and levels N_1, N_2. Suppose that $M = \gcd(N_1, N_2) > 1$ and there is a prime dividing M where both f and g are discrete series. Let ψ be the Arthur parameter of Type $(\mathbf{Y})$ corresponds to a Yoshida lift of (f, g).

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$$v_p\Big(\frac{\pi^{2k+3r} L^N(1, f, \mathrm{Ad}) L^N(1, g, \mathrm{Ad}) L^N(1, f \otimes g)}{L^N(r, f \otimes g \times \chi) L^N(r, \chi) L^N(\frac{1}{2}, f \times \Lambda) L^N(\frac{1}{2}, g \times \Lambda) \alpha_\infty \prod_{p|N} \alpha_{p,N}}\Big) > 0.$$

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Then there exists an Arthur parameter ψ_2, distinct from ψ_1 such that

$$\psi_1 \equiv \psi_2 \pmod{p} \tag{5}$$

Theorem (Saito-Kurokawa lifts)

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Then there exists an Arthur parameter ψ_2, distinct from ψ_1 such that

$$\psi_1 \equiv \psi_2 \pmod{p} \tag{6}$$

Thank you!