On Böcherer's conjecture

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Plan of today's talk

1. Böcherer's conjecture
2. Gross-Prasad conjecture
3. Refined Gross-Prasad conjecture
4. Final remark

Introduction

Definition (Siegel cusp forms of degree n)

A Siegel modular form of degree n with respect to $\mathrm{Sp}_n(\mathbb{Z})$ and weight k is a holomorphic function Φ on $\mathfrak{H}_n$ satisfying

$$\Phi\left((AZ+B)(cZ+D)^{-1}\right) = \det(CZ+D)^k \Phi(Z)$$

for any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z})$ and $Z \in \mathfrak{H}_n$. If in addition, Φ vanishes at the cusps, then Φ is called a cusp form. Here

$$\mathrm{Sp}_n(\mathbb{Z}) = \left\{\gamma \in \mathrm{GL}_{2n}(\mathbb{Z}) : {}^t\gamma J_n \gamma = J_n\right\}, \quad J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

$$\mathfrak{H}_n = \left\{Z \in \mathrm{M}_2(\mathbb{C}) : {}^tZ = Z, \mathrm{Im}(Z) > 0\right\}.$$

We denote by S_k^n the space of Siegel cusp forms of degree n with respect to $\mathrm{Sp}_n(\mathbb{Z})$ and weight k.

Fourier coefficients of Hecke eigenforms

Let $\mathcal{H}_n$ denote the Hecke algebra which acts on S_k^n.
Since $\mathcal{H}_n$ is commutative and $T \in \mathcal{H}_n$ is Hermitian with respect to the Petersson inner product $\langle \ , \ \rangle$, S_k^n has a basis consisting of Hecke eigenforms.

$\underline{n = 1}$
Let $\phi \in S_k^1$ be a Hecke eigenform with the Fourier expansion

$$\phi(z) = \sum_{n=1}^{\infty} a(n, \phi) \exp\left(2\pi\sqrt{-1}\, nz\right).$$

Then $a(1, \phi) \neq 0$ and when we normalize ϕ so that $a(1, \phi) = 1$, we have

$$a(n, \phi) = \lambda(n) \quad \textit{where } T(n)\,\phi = \lambda(n)\,\phi \textit{ and}$$

$$\sum_{n \geq 1} a(n, \phi)\, n^{-s} = \sum_{n \geq 1} \lambda(n)\, n^{-s} = \prod_{p:prime} \frac{1}{\left(1 - \alpha_p(\phi)\, p^{-s}\right)\left(1 - \beta_p(\phi)\, p^{-s}\right)}.$$

Degree 2 case

Let $\Phi \in S_k^2$ be a Hecke eigenform and assume that k is even. Then Φ has the Fourier expansion

$$\Phi(Z) = \sum_{T \in \mathcal{P}_2(\mathbb{Z})} a(T, \Phi) \exp\left(2\pi\sqrt{-1}\,\mathrm{tr}(TZ)\right)$$

where

$$\mathcal{P}_2(\mathbb{Z}) := \left\{ T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} : a, b, c \in \mathbb{Z},\ T \text{ is positive definite} \right\}.$$

For $S, T \in \mathcal{P}_2(\mathbb{Z})$, we say $S \sim T$ when $T = {}^t\gamma S \gamma$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$S \sim T \quad \Rightarrow \quad a(S, \Phi) = a(T, \Phi).$$

Relation between Hecke eigenvalues and Fourier coefficients

Suppose that $-\det(2T) = -D$ is a discriminant of $E = \mathbb{Q}\left(\sqrt{-D}\right)$.

- *Recall the Gauss composition law for binary quadratic forms.*
- *Let h_E be the class number of the imaginary quadratic fields E.*
- *Let $\{T_i : 1 \leq i \leq h_E\}$ be a set of representatives of equivalence classes of $T \in \mathcal{P}_2(\mathbb{Z})$ with $-\det(2T) = -D$.*

Definition (Bessel period of type (E, χ))

For a character χ of the ideal class group of E,

$$B(\Phi; E, \chi) := w_E^{-1} \sum_{i=1}^{h_E} \chi(T_i)\, a(T_i, \Phi) \qquad \textit{Bessel period of type } (E, \chi)$$

where w_E denotes the number of roots of unity in E.

Let $\lambda(m)$ be the eigenvalue of $T(m) \in \mathcal{H}_2$ for Φ, i.e. $T(m)\Phi = \lambda(m)\Phi$. Then:

$$\sum_{m \geq 1} \lambda(m)\, m^{-s} = \zeta(2s - 2k + 4)^{-1}\, L(s, \Phi, \mathrm{spin})$$

where $L(s, \Phi, \mathrm{spin})$ is the spin L-function of degree 4.

Then for any character χ of the ideal class group of E, we have

$$L(s - k + 2, \chi) \cdot \sum_{i=1}^{h_E} \left(\sum_{n \geq 1} a(nT_i, \Phi)\, n^{-s} \right) = w_E \cdot B(\Phi; E, \chi)\, L(s, \Phi, \mathrm{spin}).$$

In particular, when the class number $h_E = 1$, we have

$$\sum_{n \geq 1} a(nT, \Phi)\, n^{-s} = a(T, \Phi) \sum_{m \geq 1} \lambda(m)\, m^{-s}.$$

Böcherer's conjecture

Conjecture (Böcherer, circa 1986)

There exists a constant C_Φ which depends only on Φ such that

$$|B(\Phi; E)|^2 = C_\Phi \cdot D^{k-1} \cdot L\left(\frac{1}{2}, \pi(\Phi) \times \chi_E\right)$$

for any imaginary quadratic field E where

$$B(\Phi; E) := B(\Phi; E, 1) = w_E^{-1} \sum_{i=1}^{h_E} a(T_i, \Phi)$$

is the Special Bessel period of type E of Φ,

- $\pi(\Phi)$ *is the automorphic representation of* $\mathrm{PGSp}_2(\mathbb{A})$ *attached to* Φ
- χ_E *is the quadratic chracter of* $\mathbb{A}^\times$
- $L(s, \pi(\Phi) \times \chi_E)$ *is the complete spin L-function normalized so that the functional equation is with respect to* $s \mapsto 1 - s$.

Remark

- *Böcherer verified the conjecture for Saito-Kurokawa lifts.*
- *Böcherer did not speculate on the nature of the constant C_Φ.*
- *In particular, the equality above implies that*

$$B(\Phi; E) \neq 0 \Rightarrow L(1/2, \pi(\Phi) \times \chi_E) \neq 0.$$

Some natural questions arise:

- *Can we describe the constant C_Φ explicitly?*
- *Is there such a formula also for $B(\Phi; E, \chi)$ when χ is a non-trivial character? . . . Generalized Böcherer conjecture*

Gross-Prasad conjecture and its refinement

- F: a number field
- $(V, (\ ,\ )_V)$: a non-degenerate quadratic space over F.
- W: a non-degenerate subspace of V such that $\dim V \not\equiv \dim W \pmod 2$.
- (Π, V_Π): cuspidal representation of $\mathrm{SO}(V, \mathbb{A})$.
- (π, V_π): cuspidal representation of $\mathrm{SO}(W, \mathbb{A})$.

Then a certain period integral $B_{V,W} : V_\Pi \times V_\pi \to \mathbb{C}$, called Bessel period, is defined.

Conjecture (Gross-Prasad, co-dimension one case (1992), co-dimension general (1994), Canad. J. Math.)

- $B_{V,W} \not\equiv 0 \Rightarrow L(1/2, \Pi \times \pi) \neq 0$.
- *The opposite direction, i.e. the non-vanishing of the L-value implies the non-vanishing of a related Bessel period.*
- *Local counterparts.*

Conjecture (Ichino-Ikeda, 2010, GAFA)

They formulated a refinement of the Gross-Prasad conjecture, which gives an explicit identity between $B_{V,W}(\,,\,)$ *and* $L(1/2, \Pi \times \pi)$ *when* $B_{V,W}(\,,\,) \not\equiv 0$, *and* $\dim W = \dim V - 1$.

Conjecture (Generalization of Ichino-Ikeda conjecture by Yifeng Liu, 2016, Crelle)

By overcoming the convergence issue, he succeeded in formulating the generalization of the Ichino-Ikeda conjecture to the general co-dimension case.

Recall that $\mathrm{PGSp}_2 \simeq \mathrm{SO}(3,2)$.

Theorem (Dickson, Pitale, Saha & Schmidt, to appear in JMSJ)

By computing the ramified local integrals explicitly, they have shown that Yifeng Liu's conjecture for $\mathrm{SO}(3,2) \supset \mathrm{SO}(2)$ *yields a refinement of the (generalized) Böcherer conjecture in the square-free level case when* Φ *is not a Saito-Kurokawa lift.*

In particular, Dickson, Pitale Saha & Schmidt have shown that if Yifeng Liu's conjecture holds, then the constant C_Φ in Böcherer's conjecture is given by

$$C_\Phi = \frac{2^{4k-4} \cdot \pi^{2k+1}}{(2k-2)!} \cdot \frac{L(1/2, \pi(\Phi))}{L(1, \pi(\Phi), \mathrm{Ad})} \cdot \langle \Phi, \Phi \rangle .$$

Hence

$$\frac{|B(\Phi; E)|^2}{\langle \Phi, \Phi \rangle} = D^{k-1} \cdot 2^{2k-5} \cdot \frac{L(1/2, \pi(\Phi)) L(1/2, \pi(\Phi) \times \chi_E)}{L(1, \pi(\Phi), \mathrm{Ad})}.$$

F & Morimoto have shown that Yifeng Liu's conjecture holds under some conditions for the special Bessel models concerning $\mathrm{SO}_{n,n+1} \supset \mathrm{SO}(2)$. As its consequence, a refined form of Böcherer's conjecture holds by the result above of Dickson, Pitale, Saha & Schmidt.

Notation

- F: a number field.
- E: a quadratic extension of F.
- χ_E: quadratic character of $\mathbb{A}^\times / F^\times$ corresponding to E.

- All global L-functions are complete L-functions.
- $\xi_F = \prod_{v:\ all} \zeta_{F_v}(s)$: complete Dedekind zeta of F.

- $(V, \langle\ ,\ \rangle)$: a quadratic space such that $\dim V = 2n + 1 \quad (n \geq 2)$,

$$V = \mathbb{H}^{n-1} \oplus L \quad \text{(orthogonal sum) with } \mathbb{H}\text{: hyperbolic plane}$$

  and

$$\dim L = 3, \quad L \supset \left(E, \mathrm{N}_{E/F}\right) \text{ as quadratic spaces.}$$

- $\mathcal{G}_n :=$ F-isomorphism classes of $\mathrm{SO}(V)$ for such V.
- We identify $\mathrm{SO}(V)$ with its F-isomorphism class in $\mathcal{G}_n$.
- We specify $\mathbb{G} = \mathbb{G}_n = \mathrm{SO}(\mathbb{V}_n) \in \mathcal{G}_n$ to denote the *split* one.

Bessel subgroup

For $G = \mathrm{SO}(V) \in \mathcal{G}_n$, we have $\mathrm{SO}(E) \subset G$.
But $\mathrm{SO}(E)$ is "too small."

Definition (Bessel subgroup)

Taking a certain unipotent subgroup S, a Bessel subgroup R_E is defined by

$$R_E := T_E \ltimes S \quad \textit{with } T_E := \mathrm{SO}(E),$$

which is contained in a maximal parabolic subgroup of G whose Levi component is $\mathrm{GL}(n-1) \times \mathrm{SO}(L)$.

For a non-trivial character $\psi : \mathbb{A}/F \to \mathbb{C}^{\times}$, we have a character on $S(\mathbb{A})$ also denoted by ψ , by abuse of notation, which is stable under the conjugate action of $T_E(\mathbb{A})$.

Bessel period & Special Bessel period

Definition (Bessel period)

Let χ be a character of $T_E(\mathbb{A})/T_E(F)$. Note: $T_E \simeq E^\times/F^\times$.
Then for an automorphic form ϕ on $\mathrm{SO}(V,\mathbb{A})$, $B_{E,\chi,\psi}(\phi)$, a Bessel period of type (E,χ,ψ) is defined by

$$B_{E,\chi,\psi}(\phi) := \int_{T_E(F)\backslash T_E(\mathbb{A})} \int_{S(F)\backslash S(\mathbb{A})} \phi(ts)\,\chi^{-1}(t)\,\psi^{-1}(s)\,dt\,ds.$$

Definition (Special Bessel period)

When χ is trivial, the Bessel period of type $(E,1,\psi)$ is called the special Bessel period of type E and denoted by $B_E(\phi)$, i.e.

$$B_E(\phi) := \int_{T_E(F)\backslash T_E(\mathbb{A})} \int_{S(F)\backslash S(\mathbb{A})} \phi(ts)\,\psi^{-1}(s)\,dt\,ds.$$

Gross-Prasad conjecture for special Bessel periods

Theorem (F & Morimoto, Math. Ann., 2017)

- $\pi = \otimes_v \pi_v$: *an irreducible cuspidal automorphic representation of* $G(\mathbb{A})$ *for* $G \in \mathcal{G}_n$. *Let* V_π *be its space of automorphic forms.*
- *Assume that a local component* π_w *at some finite place* w *is generic.*

Suppose that $B_E \not\equiv 0$ *on* V_π.
Then

$$L(1/2, \pi)\, L(1/2, \pi \times \chi_E) \neq 0.$$

Moreover:

- $\exists\, \pi^\circ$: *globally generic irreducible cuspidal automorphic representation of* $\mathbb{G}(\mathbb{A})$ *which is nearly equivalent to* π, *i.e.* $\pi_v^\circ \simeq \pi_v$ *for almost all* v.

Dihua Jiang & Lei Zhang: recently proved a more general theorem assuming the extension of Arthur's result to the non quasi-split case.

Theorem above follows from the following theorem.

Theorem (F & Morimoto, Math. Ann., 2017)

- π: *an irreducible cuspidal automorphic representation of* $G(\mathbb{A})$ *with* $G \in \mathcal{G}_n$. *Suppose that* $B_E \not\equiv 0$ *on* V_π.

Suppose moreover that:

- $\sigma := \Theta_n(\pi, \psi)$: *theta lift of* π *from* G *to* $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ *with respect to* ψ,
- $\mathit{\Pi} := \Theta_{\mathbb{V}_n}(\sigma, \psi^{-\lambda})$: *theta lift of* σ *to* $\mathbb{G}_n(\mathbb{A})$ *with respect to* $\psi^{-\lambda}$

are both non-zero and cuspidal. Note $E = F\left(\sqrt{-\lambda}\right)$ *and* $\psi^a(x) = \psi(ax)$. *Then we have:*

$$L(1/2, \pi)\, L(1/2, \pi \times \chi_E) \neq 0$$

and $\exists\, \pi^\circ$: *globally generic irreducible cuspidal automorphic representation of* $\mathbb{G}_n(\mathbb{A})$ *nearly equivalent to* π.

Remark

This line of thought concerning special Bessel periods goes back to Waldspurger $(n = 1)$ *and Piatetski-Shapiro & Soudry* $(n = 2)$.

As a corollary, we have shown that:

Theorem (F-Morimoto, Math. Ann., 2017)

For a Hecke eigenform $\Phi \in S_k^2$, we have

$$B(\Phi; E) \neq 0 \quad \Longleftrightarrow \quad L(1/2, \pi(\Phi)) \, L(1/2, \pi(\Phi) \times \chi_E) \neq 0.$$

Actually the equivalence is proved for full modular vector valued Siegel cusp forms.

Refined Gross-Prasad conjecture

Set Up

- π: *an irreducible* *tempered* *cuspidal automorphic representation of* $G(\mathbb{A})$ *with* $G \in \mathcal{G}_n$.
- *All global measures are* *Tamagawa measures.*
- $\langle \phi_1, \phi_2 \rangle := \displaystyle\int_{G(F)\backslash G(\mathbb{A})} \phi_1(g)\, \overline{\phi_2(g)}\, dg$, *Petersson product on* V_π.
- $\langle \ , \ \rangle_v$: G_v*-invariant Hermitian inner product on* V_{π_v} *such that*

$$\langle \phi_1, \phi_2 \rangle = \prod_v \langle \phi_{1,v}, \phi_{2,v} \rangle_v \quad \text{for } \phi_i = \otimes_v \phi_{i,v} \in V_\pi.$$

Set Up (continued)

- dg_v: *measure on* G_v *such that* $\mathrm{Vol}\,(K_v, dg_v) = 1$ *for almost all* v.
- dt_v: *similarly taken measure on* $T_{E,v} := \mathrm{SO}\,(E)_v$.
- *Haar measure constants:* $dg = C_G \cdot \prod_v dg_v$, $dt = C_E \cdot \prod_v dt_v$.
- *Local integral* $\alpha_v\,(\phi_v, \phi'_v)$:

$$\alpha_v\,(\phi_v, \phi'_v) := \int_{T_{E,v}} \int_{S_v}^{\mathrm{st}} \left\langle \pi_v\,(s_v t_v)\,\phi_v, \phi'_v \right\rangle_v \psi_v^{-1}\,(s)\; dt_v\, ds_v.$$

Here $\displaystyle\int_{S_v}^{\mathrm{st}}$ *denotes the stable integration on* S_v *defined by Liu.*

- *Liu showed that when* v *is "good," we have*

$$\alpha_v\,(\phi_v, \phi'_v) = \frac{L\left(\frac{1}{2}, \pi_v\right) L\left(\frac{1}{2}, \pi_v \times \chi_{E,v}\right) \prod_{j=1}^{n} \zeta_{F_v}\,(2j)}{L\,(1, \pi_v, \mathrm{Ad})\, L\,(1, \chi_{E,v})}.$$

Theorem (F & Morimoto, J. Eur. Math. Soc. (JEMS), to appear)

- F: *totally real number field.*
- $\pi = \otimes_v \pi_v$: *irreducible cuspidal tempered automorphic representation of $G(\mathbb{A})$ for $G \in \mathcal{G}_n$.*
- *At any archimedean place v, π_v is a discrete series representation.*

Suppose that $B_E \not\equiv 0$ on V_π.
Then:

- *For any v, $\exists\, \phi'_v \in V_{\pi_v}$: $K_{G,v}$-finite vector such that $\alpha_v(\phi'_v, \phi'_v) \neq 0$.*
- *For any non-zero $\phi \in V_\pi$ of the form $\phi = \otimes_v \phi_v$, we have*

$$\frac{|B_E(\phi)|^2}{\langle \phi, \phi \rangle} = 2^{-\ell} C_E \times \frac{L\left(\frac{1}{2}, \pi\right) L\left(\frac{1}{2}, \pi \times \chi_E\right) \prod_{j=1}^{n} \xi_F(2j)}{L(1, \pi, \mathrm{Ad})\, L(1, \chi_E)} \cdot \prod_v \frac{\alpha_v^\natural(\phi_v, \phi_v)}{\langle \phi_v, \phi_v \rangle}.$$

(Recall that all L-functions are complete L-functions.)

(Theorem continued) Here

$$\alpha_v^\natural(\phi_v, \phi_v) := \frac{L(1, \pi_v, \mathrm{Ad})\, L(1, \chi_{E,v})}{L(1/2, \pi_v)\, L(1/2, \pi_v \times \chi_{E,v}) \prod_{j=1}^n \zeta_{F_v}(2j)} \cdot \alpha_v(\phi_v, \phi_v)$$

and hence $\frac{\alpha_v^\natural(\phi_v,\phi_v)}{\langle \phi_v, \phi_v \rangle_v} = 1$ *for almost all* v.

- π *has a weak lift* Π *to* $\mathrm{GL}_{2n}(\mathbb{A})$, *i.e.* $\Pi = \otimes_v \Pi_v$ *is an irreducible automorphic representation of* $\mathrm{GL}_{2n}(\mathbb{A})$ *such that* Π_v *is a local Langlands lift of* π_v *at all archimedean and almost all non-archimedean* v. *Then* Π *is of the form* $\Pi = \boxplus_{i=1}^{\ell} \pi_i$ *(isobaric sum) such that*
  - π_i: *irreducible cuspidal automorphic representation of* $\mathrm{GL}_{2n_i}(\mathbb{A})$ *such that* $L(s, \pi_i, \wedge^2)$ *has a pole at* $s = 1$, $\sum_{i=1}^k n_i = n$, $\pi_i \not\simeq \pi_j$ $(i \neq j)$.

  (Indeed the existence of such Π *readily follows from the previous theorem.)*

When $n = 2$, Theorem above has been proved by Liu for endoscopic Yoshida lifts and by Corbett for non-endoscopic Yoshida lifts.

Skeleton of the proof of Theorem

First:

We define the generic character ψ_λ for $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ by

$$\psi_\lambda\left[\begin{pmatrix} u & \\ & {}^t u^{-1}\end{pmatrix}\begin{pmatrix} 1_n & S \\ & 1_n\end{pmatrix}\right] := \psi\left(u_{1,2} + \cdots + u_{n-1,n} + \frac{\lambda}{2}s_{n,n}\right).$$

Then:

Definition (ψ_λ-Whittaker period)

For an automorphic form $\tilde{\phi}$ on $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$,

$$W\left(\tilde{\phi}, \psi_\lambda\right) := \int_{U_n(F)\backslash U_n(\mathbb{A})}\int_{V_n(F)\backslash V_n(\mathbb{A})} \psi_\lambda\left[\begin{pmatrix} u & \\ & {}^t u^{-1}\end{pmatrix}\begin{pmatrix} 1_n & S \\ & 1_n\end{pmatrix}\right]$$
$$\times\, \phi\left[\begin{pmatrix} u & \\ & {}^t u^{-1}\end{pmatrix}\begin{pmatrix} 1_n & S \\ & 1_n\end{pmatrix}\right] du\, dS.$$

In the following, $A \underset{\text{a.a.}}{=} B$ implies that $A = B$ *up to multiplication by a product of finitely many local factors*.

1. Global pull-back formula of Bessel periods by F (Crelle, 1995):

$$W(\tilde{\phi};\psi_\lambda) \underset{\text{a.a.}}{=} C_G C_E^{-1} \cdot B_E(\phi) \quad \text{where } \tilde{\phi} := \theta_\psi^\varphi(\phi).$$

2. Explicit formula for metaplectic Whittaker periods by Lapid-Mao:

$$\frac{|W(\tilde{\phi};\psi_\lambda)|^2}{< \tilde{\phi},\tilde{\phi} >} \underset{\text{a.a.}}{=} 2^{-\ell} \cdot \frac{L(1/2,\pi\times\chi_E)\prod_{j=1}^n \xi_F(2j)}{L(1,\pi,\mathrm{Ad})}.$$

3. Precise Rallis inner product formula by Gan-Takeda:

$$\frac{< \tilde{\phi},\tilde{\phi} >}{< \phi,\phi >} \underset{\text{a.a.}}{=} C_G \cdot \frac{L(1/2,\pi)}{\prod_{j=1}^n \xi_F(2j)}.$$

$\Longrightarrow$ We are reduced to proving a *pull-back formula for the local metaplectic Whittaker pairing*.

Final remark

(F-Morimoto, in preparation)

1. *Generalization to $B(\Phi; E, \chi)$ where χ is not necessarily trivial of the explicit L-value formula conjectured by Dickson, Pitale, Saha & Schmidt.*
2. *Generalization of the L-value formula to the vector valued case.*
3. *Generalization of the L-value formula to the weight two case.*