Critical L-values and congruences for Siegel modular forms, I

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A theorem of Shimura

Theorem (Shimura)

Let $f \in S_k(N)$ *be a primitive Hecke eigenform,* χ *a Dirichlet character. Then there exist* $0 \neq u(\epsilon, f) \in \mathbb{C}, \epsilon \in \{0, 1\}$*, such that, for every* $\sigma \in \mathrm{Aut}(\mathbb{C})$ *and* $0 < m < k$*, we have*

$$\sigma\Big(\frac{L(m, f, \chi)}{(2\pi i)^m G(\chi) u(\epsilon, f)}\Big) = \frac{L(m, f^\sigma, \chi^\sigma)}{(2\pi i)^m G(\chi^\sigma) u(\epsilon, f^\sigma)}$$

with $(-1)^\epsilon = (-1)^m \chi(-1)$*.*

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Algebraicity of special values of $L(s, f \boxtimes \chi, \mathrm{sym}^n)$.

- $n = 1$: Shimura
- $n = 2$: Sturm
- $n = 3$: Garrett, Harris
- n odd: Raghuram (assuming functoriality of sym^n)

Algebraicity of critical values of sym^4

Theorem (P., Saha, Schmidt)

Let $k \geq 2$ be even. Let τ be a cuspidal, non-dihedral, automorphic representation on $\mathrm{PGL}_2(\mathbb{A})$ with τ_∞ isomorphic to the holomorphic discrete series representation of lowest weight k. Let χ be an odd Dirichlet character and r be an odd integer such that $1 \leq r \leq k-1$. Furthermore, if $\chi^2 = 1$, we assume that $r \neq 1$. Then there exists a real number $C(\tau)$ such that, for any finite subset S of places of $\mathbb{Q}$ that includes the archimedean place, and for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\sigma\Big(\frac{L^S(r, \chi \otimes \mathrm{sym}^4\tau)}{(2\pi i)^{3r} G(\chi)^3 C(\tau)}\Big) = \frac{L^S(r, \chi^\sigma \otimes \mathrm{sym}^4(\tau^\sigma))}{(2\pi i)^{3r} G(\chi^\sigma)^3 C(\tau^\sigma)}.$$

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Ramakrishnan-Shahidi: $\tau \mapsto F_\tau$ a vector-valued holomorphic Siegel cusp form of degree 2 satisfying

$$L(s,\mathrm{sym}^4\tau) = L(s,F_\tau,\mathrm{std}).$$

Algebraicity of critical values of standard L-function

Theorem (P., Saha, Schmidt)

Let π be a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ such that π_∞ is isomorphic to the holomorphic discrete series representation with highest weight (k_1, k_2) such that $k_1 \geq k_2 \geq 3, k_1 \equiv k_2 \pmod 2$. Let χ be any Dirichlet character satisfying $\chi(-1) = (-1)^{k_1}$. Let r be any integer satisfying $1 \leq r \leq k_2 - 2, r \equiv k_2 \pmod 2$. Furthermore, if $\chi^2 = 1$, we assume that $r \neq 1$. Let S be any finite subset of places of $\mathbb{Q}$ that includes the archimedean place. Then there exists $0 \neq C(\pi) \in \mathbb{C}$ such that, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\sigma\Big(\frac{L^S(r, \pi \boxtimes \chi, \mathrm{std})}{(2\pi i)^{2k+3r} G(\chi)^3 C(\pi)}\Big) = \frac{L^S(r, \pi^\sigma \boxtimes \chi^\sigma, \mathrm{std})}{(2\pi i)^{2k+3r} G(\chi^\sigma)^3 C(\pi^\sigma)}$$

Previous results

Classically, theorem applies to vector valued Siegel cusp forms of weight $\det^{k_2} \mathrm{sym}^{k_1-k_2}$ with respect to an arbitrary congruence subgroup of $\mathrm{Sp}(4,\mathbb{Q})$. Previous known results:

- For $k_1 = k_2$ scalar case : by Shimura but only for $\Gamma_0(N)$-type congruence subgroups.
- For $k_1 > k_2$ vector valued case : by Kozima but only for full level and $\chi = 1$.

A pullback formula by Garrett

Consider the Siegel upper half space of genus n defined by

$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) : Z = {}^{\mathrm{t}}Z, \mathrm{Im}(Z) > 0\}.$$

For $Z \in \mathbb{H}_n$, $s \in \mathbb{C}$, define the Eisenstein series by

$$E_{n,k}(Z,s) := \sum_{C,D} \det(\mathrm{Im}(Z))^s \det(CZ+D)^{-k} |\det(CZ+D)|^{-s}.$$

We have $E_{n,k}(Z,0) \in M_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$.

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$$\int_{\mathrm{Sp}(2n,\mathbb{Z})\backslash\mathbb{H}_n} F(-\bar{Z}_1) E_{2n,k}(\begin{bmatrix} Z_1 & \\ & Z_2 \end{bmatrix}, s) dZ_1$$

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Want to extend this to i) incorporate characters, ii) include arbitrary congruence subgroups, and iii) cover the case of vector valued Siegel cusp forms.

Adelic formulation

- $G_{4n} := \mathrm{GSp}(4n)$ and let P_{4n} Siegel parabolic subgroup of matrices whose lower left $(2n) \times (2n)$ block is zero.
- The Levi subgroup of P_{4n} is $\mathrm{GL}(2n) \times \mathrm{GL}(1)$ given by matrices of the form $\begin{bmatrix} A & \\ & v\,{}^tA^{-1} \end{bmatrix}$, with $A \in \mathrm{GL}(2n)$ and $v \in \mathrm{GL}(1)$.
- Let χ be a character of $\mathbb{A}^\times$. Set $I(\chi, s) = \mathrm{Ind}_{P_{4n}(\mathbb{A})}^{G_{4n}(\mathbb{A})}(\chi \delta_{P_{4n}}^s)$. Here, χ acts on $\begin{bmatrix} A & \\ & v\,{}^tA^{-1} \end{bmatrix}$ by $\chi(v^{-n}\det(A))$. For a smooth section $f(\cdot, s) \in I(\chi, s)$, define

$$E(g, s, f) := \sum_{\gamma \in P_{4n}(F) \backslash G_{4n}(F)} f(\gamma g, s).$$

- Let (π, V_π) be a cuspidal automorphic representation of $G_{2n}(\mathbb{A})$ and let $\phi \in V_\pi$.

Zeta integral

Fix an embedding of $H_{2n,2n}$ of $G_{2n} \times G_{2n}$ in G_{4n}. Define

$$Z(s; f, \phi)(g) = \int\limits_{\mathrm{Sp}_{2n}(F)\backslash g\cdot\mathrm{Sp}_{2n}(\mathbb{A})} E((h, g), s, f)\phi(h)dh.$$

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Unwinding the integral, we get

$$Z(s; f, \phi)(g) = \int\limits_{\mathrm{Sp}_{2n}(\mathbb{A})} f(Q_n \cdot (h, 1), s)\phi(gh)dh.$$

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Theorem (Basic Identity)

Assume that ϕ is a pure tensor $\otimes_v \phi_v$ and f factors into $\otimes f_v$ with $f_v \in I(\chi_v, s)$. Then $Z(s; f, \phi)$ also belongs to V_π and corresponds to the pure tensor $\otimes_v Z_v(s; f_v, \phi_v)$, where

$$Z_v(s; f_v, \phi_v) = \int\limits_{\mathrm{Sp}_{2n}(F_v)} f_v(Q_n \cdot (h, 1), s)\pi_v(h)\phi_v dh \in \pi_v.$$

The local integral

Goal: Want to choose f_v and ϕ_v such that

$$Z_v(s; f_v, \phi_v) = \int\limits_{\mathrm{Sp}_{2n}(F_v)} f_v(Q_n \cdot (h,1), s)\pi_v(h)\phi_v dh = B_v(s)\phi_v.$$

The unramified computation

The **unramified** computation follows as in PS-Rallis or Bocherer. When both χ_v and π_v are unramified, we choose f_v and ϕ_v to be the unramified vectors. Then we get that $Z_v(s; f_v, \phi_v)$ is equal to

$$\frac{L((2n+1)s + 1/2, \pi_v \boxtimes \chi_v, \varrho_{2n+1})}{L((2n+1)(s+1/2), \chi_v) \prod_{i=1}^{n} L((2n+1)(2s+1) - 2i, \chi_v^2)} \phi_v.$$

The ramified computation

- Choose a positive integer m such that $\chi_v|_{(1+\mathfrak{p}^m)\cap\mathfrak{o}^\times} = 1$ and π_v has a vector ϕ_v fixed by $\Gamma_{2n}(\mathfrak{p}^m)$.

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- Choose ϕ_v to be the above vector. And choose f_v to be the unique function on $G_{4n}(F_v) \times \mathbb{C}$ with support $P_{4n}(F_v)Q_n\Gamma_{4n}(\mathfrak{p}^m)$, given by

$$f_v(pQ_nk) = \chi_v(p)\delta_{P_{4n}}(p)^{s+\frac{1}{2}}, p \in P_{4n}(F_v), k \in \Gamma_{4n}(\mathfrak{p}^m).$$

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- With these choices, we have

$$Z_v(s; f_v, \phi_v) = \mathrm{vol}(\Gamma_{2n}(\mathfrak{p}^m))\phi_v.$$

Archimedean computation

Assumption: Let π_∞ be a holomorphic discrete series with minimal K-type of highest weight $\mathbf{k} = (k_1, \cdots, k_n)$ with integers $k_1 \geq k_2 \geq \cdots \geq k_n \geq n$ and all the k_i have the same parity. Set $k = k_1$. Assume that $\chi_\infty = \mathrm{sgn}^k$.

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- Choose $f_k \in I(\mathrm{sgn}^k, s)$ to be the scalar valued weight k vector.
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- One can show that one dimensional K-type with highest weight $(k, k, \cdots, k)$ occurs exactly once in π_∞. Choose $\phi_\infty = \phi_{\mathbf{k}}$ the vector spanning this one dimensional space.
- It turns out that $Z(s, f_k, \phi_{\mathbf{k}})$ is also a weight k vector in π_∞. Hence,

$$Z_\infty(s, f_k, \phi_{\mathbf{k}}) = \int\limits_{\mathrm{Sp}_{2n}(\mathbb{R})} f_k(Q_n \cdot (h, 1), s)\pi_\infty(h)\phi_{\mathbf{k}} dh = B_{\mathbf{k}}(s)\phi_{\mathbf{k}}.$$

The scalar minimal K-type

Taking inner product with $\phi_{\mathbf{k}}$, we get

$$\langle \int\limits_{\mathrm{Sp}_{2n}(\mathbb{R})} f_k(Q_n \cdot (h,1), s)\pi_\infty(h)\phi_{\mathbf{k}} dh, \phi_{\mathbf{k}}\rangle = B_{\mathbf{k}}(s)\langle \phi_{\mathbf{k}}, \phi_{\mathbf{k}}\rangle.$$

Normalize $\phi_{\mathbf{k}}$ so that $\langle \phi_{\mathbf{k}}, \phi_{\mathbf{k}}\rangle = 1$. We have

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Let $k = k_1 = k_2 = \cdots = k_n$, the scalar minimal K-type case. We have an explicit formula for the matrix coefficient.

$$\langle \pi_k(h)\phi_{\mathbf{k}}, \phi_{\mathbf{k}}\rangle = \begin{cases} \dfrac{\mu_n(h)^{nk/2}\, 2^{nk}}{\det\left(A + D + i(C - B)\right)^k} & \text{for } \mu_n(h) > 0, \\ 0 & \text{for } \mu_n(h) < 0. \end{cases}$$

Here $h = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GSp}(2n, \mathbb{R})$ and μ_n is the similitude function.

The scalar minimal K-type contd.

The integrand is left and right K-invariant function. Using an integration formula involving the KAK decomposition and explicit formula for f_v, we can compute the integral to get

$$Z_\infty(s, f_k, \phi_{\mathbf{k}}) = i^{nk}\,\pi^{n(n+1)/2} C_k((2n+1)s - 1/2)\ \phi_{\mathbf{k}}$$

with the function $C_k(z)$ is defined as

$$C_k(z) = 2^{-n(z-1)}\left(\prod_{j=1}^{n}\prod_{i=1}^{j}\frac{1}{z+k-1-j+2i}\right) \times \left(\prod_{j=1}^{n}\frac{z-(k-1-j)}{z+(k-1-j)}\right).$$

The general minimal K-type case

No formula for matrix coefficients in the general case. We embed $\pi_{\mathbf{k}}$ in an induced representation where we know the explicit formula for the vector $\phi_{\mathbf{k}}$.

$$Z_\infty(s, f_k, \phi_{\mathbf{k}})(1) = \int\limits_{\mathrm{Sp}_{2n}(\mathbb{R})} f_k(Q_n \cdot (h,1), s)\pi_\infty(h)(\phi_{\mathbf{k}}(1))dh = B_{\mathbf{k}}(s)\phi_{\mathbf{k}}(1).$$

Normalizing $\phi_{\mathbf{k}}(1) = 1$, we get

$$B_{\mathbf{k}}(s) = \int\limits_{\mathrm{Sp}_{2n}(\mathbb{R})} f_k(Q_n \cdot (h,1), s)\phi_{\mathbf{k}}(h)dh$$

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"Complicated Integral" depends only on $k = k_1$ and not on $k_2, k_3, \cdots, k_n$.

The archimedean integral

$$Z_\infty(s, f_k, \phi_{\mathbf{k}}) = i^{nk}\, \pi^{n(n+1)/2} A_{\mathbf{k}}((2n+1)s - 1/2)\ \phi_{\mathbf{k}}$$

with the function $A_{\mathbf{k}}(z)$ is defined as

$$A_{\mathbf{k}}(z) = 2^{-n(z-1)} \left(\prod_{j=1}^{n} \prod_{i=1}^{j} \frac{1}{z + k - 1 - j + 2i} \right) \times \left(\prod_{j=1}^{n} \prod_{i=0}^{\frac{k-k_j}{2}-1} \frac{z - (k-1-j-2i)}{z + (k-1-j-2i)} \right).$$

Global result

Theorem (P., Saha, Schmidt)

Let $N = \prod p^{m_p}$, where $m_p = 0$ if both π_p and χ_p are unramified, and otherwise they are chosen as in previous slides. The partial function $L^N(s, \pi \boxtimes \chi, \varrho_{2n+1})$ can be analytically continued to a meromorphic function of s with only finitely many poles. Furthermore, for all $s \in \mathbb{C}$, we have the relation

$$Z(s,f,\phi)(g) = \frac{L^N((2n+1)s + 1/2, \pi \boxtimes \chi, \varrho_{2n+1})}{L^N((2n+1)(s+1/2), \chi) \prod_{j=1}^{n} L^N((2n+1)(2s+1) - 2j, \chi^2)}$$

$$\times\, i^{nk}\, \pi^{\frac{n(n+1)}{2}} \left(\prod_{p|N} \mathrm{vol}(\Gamma_{2n}(p^{m_p})) \right) A_{\mathbf{k}}((2n+1)s - \frac{1}{2}) \phi(g).$$

Classical version

Let F be the smooth function on the Siegel upper half plane $\mathbb{H}_n$ corresponding to ϕ and set

$$E^{\chi}_{k,N}(Z,s) := j(g,I)^k E(g, \frac{2s}{2n+1} + \frac{k}{2n+1} - \frac{1}{2}, f),$$

where $g \in G_{4n}(\mathbb{R})$ such that $g\langle I \rangle = Z$. Then

$$\langle E^{\chi}_{k,N}(\,-\,, Z_2, \frac{n}{2} - \frac{k-s}{2}), \bar{F}\rangle = \frac{L^N(s, \pi \boxtimes \chi, \varrho_{2n+1})}{L^N(s+n,\chi)\prod_{j=1}^{n} L^N(2s+2j-2,\chi^2)} A_{\mathbf{k}}(s-1)$$
$$\times \prod_{p|N} \mathrm{vol}(\Gamma_{2n}(p^{m_p})) \times \frac{i^{nk}\,\pi^{n(n+1)/2}}{\mathrm{vol}(\mathrm{Sp}_{2n}(\mathbb{Z})\backslash \mathrm{Sp}_{2n}(\mathbb{R}))} \times F(Z_2).$$

Nearly holomorphic Siegel modular forms in genus 2

- For integers $\ell \geq 1$ and $m \geq 0$ such that m is even, denote by $S_{\ell,m}(\Gamma_4(N))$ the space of holomorphic Siegel cusp forms with respect to $\Gamma_4(N)$ and weight $(\ell + m, \ell)$ (or $\det^{\ell} \mathrm{sym}^m$).
- Let D_+ and U be the differential operators that map cusp forms of weight $(\ell + m, \ell)$ to those with weight $(\ell + m + 2, \ell + 2)$ and $(\ell + m, \ell + 2)$ respectively.

Theorem

We have

$$N_k(\Gamma_4(N))^\circ = \bigoplus_{\substack{\ell=2 \\ \ell \equiv k \bmod 2}}^{k} \bigoplus_{\substack{m=0 \\ m \equiv 0 \bmod 2}}^{k-\ell} D_+^{(k-\ell-m)/2} U^{m/2} S_{\ell,m}(\Gamma_4(N))$$

We also obtain $\mathrm{Aut}(\mathbb{C})$-equivariance of the Petersson inner product using certain holomorphic projection operators.

Special values for $n = 2$

Theorem (P., Saha, Schmidt)

Let π be a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ such that π_∞ is isomorphic to the holomorphic discrete series representation with highest weight (k_1, k_2) such that $k_1 \geq k_2 \geq 3, k_1 \equiv k_2 \pmod 2$. Let χ be any Dirichlet character satisfying $\chi(-1) = (-1)^{k_1}$. Let r be any integer satisfying $1 \leq r \leq k_2 - 2, r \equiv k_2 \pmod 2$. Furthermore, if $\chi^2 = 1$, we assume that $r \neq 1$. Let S be any finite subset of places of $\mathbb{Q}$ that includes the archimedean place. Then, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\sigma\Big(\frac{L^S(r, \pi \boxtimes \chi, \mathrm{std})}{(2\pi i)^{2k+3r} G(\chi)^3 \langle F, F\rangle}\Big) = \frac{L^S(r, \pi^\sigma \boxtimes \chi^\sigma, \mathrm{std})}{(2\pi i)^{2k+3r} G(\chi^\sigma)^3 \langle F^\sigma, F^\sigma\rangle}$$

Thank you. And part II after coffee.