Intersection Theory

4. Exercise sheet

Exercise 1:
Let k be a field and $H \subset \mathbb{P}^n_k$ be a hypersurface. Denote by $H_1, \ldots, H_r$ the irreducible components of H, hypersurfaces themselves of degree $d_i = \deg(H_i)$. Prove that

$$\mathrm{CH}_{n-1}(\mathbb{P}^n_k \setminus H) \cong \mathbb{Z}/(d_1, \ldots, d_r).$$

Exercise 2:
Let k be a field and denote by R, the k-algebra $k[\epsilon]/(\epsilon^2)$ and P the scheme $\mathrm{Proj}\,(R[x_0, x_1]/(x_0\epsilon))$ over $\mathrm{Spec}\,k$ for the grading given by degrees of x_0, x_1.

(a) Show that $Z^1(P) = Z^1(\mathbb{P}^1_k)$.

(b) Identify the mistake in the following:

$$\mathrm{div}(t) = 2 \cdot [0] - [\infty]$$

which is not rationally equal to 0.

Exercise 3: Find an example of a variety X for which the assignment

$$S_X : U \mapsto \{\text{non-zero divisors in } \mathcal{O}_X(U)\}$$

does not form a presheaf.
In general, the correct definition of the presheaf of meromorphic functions is as follows:

$$M_X(U) = \mathcal{O}_X(U)[S(U)]^{-1}$$

where $S(U)$ denotes the functions whose restrictions to $\mathcal{O}_{X,x}$ for every $x \in U$ is a non-zero divisor.

Hint: Note that $S(U)$ agrees with $S_X(U)$ if X has no embedded components.

Exercise 4*: Find an example of a variety X and a line bundle $\mathcal{L}$ on X which does not come from a Cartier divisor on X.