Intersection Theory

4. Exercise sheet

Exercise 1:

The **exterior product** of cycles, for algebraic schemes X, Y over a field k, is the bilinear map

$$Z_*(X) \times Z_*(Y) \to Z_*(X \times Y)$$
$$(\alpha, \beta) \mapsto \alpha \boxtimes \beta$$

defined on prime cycles $[V] \in Z_r(X)$ and $[W] \in Z_s(Y)$ as

$$[V] \boxtimes [W] = [V \times W] \in Z_{r+s}(X \times Y).$$

In this exercise, we will work through various properties of the exterior product. Prove the following:

1. *Compatibility with rational equivalence:*

$$\text{If } \alpha \sim 0 \text{ or } \beta \sim 0, \text{ then } \alpha \boxtimes \beta \sim 0.$$

2. *Compatibility with proper pushforward:*
   For $f : X \to X'$, $g : Y \to Y'$ proper, the product $f \times g : X \times Y \to X' \times Y'$ is also proper and one has
   $$(f \times g)_*(\alpha \boxtimes \beta) = f_*\alpha \boxtimes g_*\beta$$
   for all $\alpha \in Z_*(X), \beta \in Z_*(Y)$.

3. *Compatibility with flat pullback:*
   For $f : X \to X'$ and $g : Y \to Y'$ flat equidimensional maps of relative dimension r and s respectively, the product $f \times g : X \times Y \to X' \times Y'$ is flat equidimensional of relative dimension $r + s$ and one has
   $$(f \times g)^*(\alpha \boxtimes \beta) = f^*\alpha \boxtimes g^*\beta$$
   for all $\alpha \in Z_*(X'), \beta \in Z_*(Y')$.

Exercise 2:

Let k be a field and X, Y algebraic schemes over k. Suppose X admits a cellular decomposition, i.e, there is filtration of X composed of closed subschemes

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n = X,$$

such that $X_i \setminus X_{i-1}$ is isomorphic to a disjoint union of affine spaces of dimension i. Prove that the exterior product of cycles gives rise to a surjective map

$$\mathrm{CH}_*(X) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Y) \to \mathrm{CH}_*(X \times Y).$$

Hint: Use that $\mathrm{CH}(\mathbb{A}^n_X) \cong \mathrm{CH}(X)$.

Exercise 3*:

The map

$$\mathrm{CH}_*(X) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Y) \to \mathrm{CH}_*(X \times Y)$$

from Exercise 2 is known to also be injective, thus an isomorphism, in many examples. Can you work out examples where it fails to be a surjection?