Intersection Theory

3. Exercise sheet

Exercise 1:

Let A be a Noetherian local ring of dimension 1 and let $f \in A$ be an element which is not a zero divisor. Prove that for all finitely generated A-modules M, the complex

$$M \xrightarrow{f\cdot} M$$

has kernel and cokernel of finite length and the Euler characteristic of it, which is defined as

$$\chi_A(f, M) := \operatorname{length}_A(\ker M \xrightarrow{f\cdot} M) - \operatorname{length}_A(M/fM)$$

is equal to

$$\chi_A(f, M) = \sum_{\operatorname{ht}(\mathfrak{p})=0} \operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) \cdot \chi_{A/\mathfrak{p}}(f, A/\mathfrak{p}).$$

Hint: Use the additivity of $\chi_A(f, M)$ on short exact sequences together with the theory of associated prime ideals of a Noetherian module to reduce to case $M = A/\mathfrak{p}$.

Exercise 2:

Let k be a field, and consider the following Cartesian square of algebraic schemes over k:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Assume that, $f : X \to Y$ is proper and $g : Y \to Y'$ is flat equidimensional of dimension d. Following the outlined steps, prove that for all $\alpha \in Z_*(X)$,

$$g^* f_* \alpha = f'_* g'^* \alpha \in Z_{*+d}(Y').$$

(i) Prove the statement for the case where X is a variety, f is a closed immersion and α is a prime cycle.

(ii) Using (i), reduce the statement to proving it for the case where X and Y are varieties.

(iii) Using (ii) and base changing to the generic point we may assume $Y = \operatorname{Spec} K$ for a field K. In this case, prove the statement for $\dim(X) - \dim(Y) > 0$.

(iv) Prove the statement for the case $Y = \operatorname{Spec} K$ and $X = \operatorname{Spec} L$ for a finite field extension L/X and conclude.