Intersection Theory

2. Exercise sheet

Exercise 1:
Give an example of a morphism $f : X \to Y$ of varieties, such that the pushforward map $f_* : Z_*(X) \to Z_*(Y)$ does not descend to $CH_*(X) \to CH_*(Y)$.

Hint: The map f cannot be proper.

Exercise 2: (Bézout's Theorem)
Let k be an algebraically closed field. Let $F, G \subset \mathbb{P}^2_k$ be two curves of degrees m and n respectively. Assume that $F \cap G$ is zero dimensional, that is, their irreducible components are pairwise distinct. Prove that

$$\sum_{P \in \mathbb{P}^2(k)} i(P, F \cdot G) = mn.$$

Hint: Use the fact that pushforward of cycles under a proper map respects rational equivalence to reduce to a simpler case.

Exercise 3:
Let k be a field and V a variety of dimension $d+1$ over k. Let $f : V \to \mathbb{P}^1_k$ be a dominant morphism. Then $f^{-1}(0), f^{-1}(\infty)$ are either empty or equidimensional subschemes of V of dimension d.

(a) Show that f determines a rational function $\tilde{f}$ on V.

(b) Prove that $\left[f^{-1}(0)\right] - \left[f^{-1}(\infty)\right] = \left[\mathrm{div}(\tilde{f})\right]$ in $Z_d(V)$.

Exercise 4:
Let k be a field and X a k-variety. For a closed subvariety V of $X \times_{\mathrm{Spec}(k)} \mathbb{P}^1_k$ which is dominant over $\mathbb{P}^1_k$, define $V_0, V_\infty \subset X$ as

$$V_0 := \left(X \times_{\mathrm{Spec}(k)} \{0\}\right) \times_{\mathbb{P}^1_X} V, \quad V_\infty := \left(X \times_{\mathrm{Spec}(k)} \{\infty\}\right) \times_{\mathbb{P}^1_X} V$$

respectively. Prove that a cycle in $Z_*(X)$ is rationally equivalent to 0 if and only if it is a sum of cycles of the form

$$[V_0] - [V_\infty].$$