Intersection Theory

1. Exercise sheet

Exercise 1:

Let X be an algebraic scheme. Show the following basic properties of Chow groups:

(1) Let X_{red} be the underlying reduced subscheme. Then, $\mathrm{CH}_*(X) = \mathrm{CH}_*(X_{\mathrm{red}})$.

(2) Let $X = \sqcup X_i$ be a finite disjoint union of open and closed subschemes. Then, $\mathrm{CH}_*(X) = \oplus \mathrm{CH}_*(X_i)$.

(3) Let $X = X_1 \cup X_2$ be a union of two closed subschemes. Then, there is an exact sequence $\mathrm{CH}_*(X_1 \cap X_2) \to \mathrm{CH}_*(X_1) \oplus \mathrm{CH}_*(X_2) \to \mathrm{CH}_*(X) \to 0$.

(4) Let $Z \subset X$ be a closed subscheme with open complement U. Then, there is an exact sequence $\mathrm{CH}_*(Z) \to \mathrm{CH}_*(X) \to \mathrm{CH}_*(U) \to 0$.

Hint for (3) and (4): Snake lemma.

Exercise 2*:

Let $X = \operatorname{Spec} A$ be a normal, affine variety of dimension $d \geq 1$. Show that A is a unique factorization domain if and only if $\mathrm{CH}_{d-1}(X) = 0$.

Hint: Use the following facts from commutative algebra: (1) A Noetherian normal domain is a unique factorization domain if and only if every prime ideal of height 1 is principal. (2) In a Noetherian normal domain A one has $A = \cap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$.

Exercise 3:

Let k be a field. Compute $\mathrm{CH}_*(X)$ in the following cases:

(1) $X = \mathbb{A}^1_k$

(2) $X = \mathbb{P}^1_k$

In addition, show the first map in the sequences in Exercise 1, (3) and (4) is not exact in general.

Hint for (2): Let $f \in k(t)^\times$ be an element in the function field of $\mathbb{P}^1_k$. Write $\mathrm{div}(f) = \sum_{i \in I} n_i [x_i] \in Z_0(\mathbb{P}^1_k)$. Then, $\sum_{i \in I} n_i = 0$.

Exercise 4:

Let k be a field. Let $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$ be the affine line over k. Let $\mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$ be the affine plane over k. Consider the following two cases:

(i) Let $X := V(y^2 - x^3) \subset \mathbb{A}^2_k$ be the *cuspidal curve*, and $f \colon \mathbb{A}^1_k \to X$ be the morphism induced from $x \mapsto t^2$, $y \mapsto t^3$ on rings.

(ii) Let $X := V(y^2 - x^2(x + 1)) \subset \mathbb{A}^2_k$ be the *nodal curve*, and let $f \colon \mathbb{A}^1_k \to X$ be the morphism induced from $x \mapsto t^2 - 1$, $y \mapsto t(t^2 - 1)$ on rings.

Then X is an affine variety of dimension 1. Work on the following statements:

(1) The map f is the normalization of X in its function field K. It restricts to an isomorphism $\mathbb{A}^1_k \backslash f^{-1}(P) \cong X \backslash \{P\}$ for $P := (0, 0) \in X(k)$. The local ring $\mathcal{O}_{X,P}$ is not normal.

(2) Compute $\mathrm{ord}_P(\frac{y}{x})$ on X where $\frac{y}{x} \in K^\times$.

(3)* Can you compute $\mathrm{CH}_0(X)$?