Intersection Theory

6. Exercise sheet

Exercise 1:
Let k be a field and D_1, D_2 two effective Cartier divisors on $X = \mathbb{P}^2_k$. Denote by

$$\sigma : \tilde{X} \to X$$

the blow-up in the ideal sheaf defined by $D_{12} = D_1 \cap D_2$. Recall the following statement proved in the lecture:

(a) The subscheme $E = \sigma^{-1}(D_1 \cap D_2)$ is an effective Cartier divisor on $\tilde{X}$.

(b) There are disjoint effective Cartier divisors C_1 and C_2 on $\tilde{X}$ such that for $i = 1, 2$,

$$\sigma^* D_i = E + C_i.$$

Consider the following three situations:

(1) One has $D_1 \subset D_2$ as subschemes.

(2) Let L_i for $i = 1, 2, 3$ be lines in generic position with respect to each other and $D_i = L_i + L_3$ for $i = 1, 2$.

(3) With notation as in (2), $D_1 = 2L_3 + L_1$ and $D_2 = L_2 + L_3$.

In each case, identify C_1, C_2 and verify (b) through explicit calculation.

Exercise 2:
If X/k is a smooth projective surface, then all Weil divisors are Cartier divisors. Thus we may define the intersection pairing

$$\begin{aligned} Z_1(X) \times Z_1(X) &\to \mathrm{CH}_0(X) \stackrel{\deg}{\to} \mathbb{Z} \\ (D_1, D_2) &\mapsto (D_1 \cdot D_2) = \deg(D_1 \cdot [D_2]). \end{aligned}$$

Using the results proven in the last lecture:

(a) Check that this is well defined and commutative.

(b) Compare this with the known intersection pairing (from the introductory lecture).

(c) If $X = \mathbb{P}^2_k$ prove that it descends to CH_1. Moreover prove using (b) that it recovers Bezout's theorem.