THETA-FUNCTIONS AND LIFTS
INTRODUCTORY TALK ENTR-WORKSHOP

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Abstract. Theta functions play an essential role in many areas of number theory. An exciting application is their use for the functional equation of the Riemann zeta function. Moreover, theta functions are deeply intertwined with the theory of quadratic forms and their arithmetic properties.

A theta lift can be constructed by integrating automorphic forms against a theta function with similar automorphic properties. A result is again an automorphic form whose Fourier expansion encodes arithmetic information related to the input automorphic form.

The goal of the talk is to informally introduce the concept of theta lift and explain it with the help of an example.

1. Motivation

Theta series play an important role in the study of quadratic forms. Let $Q\colon \mathbb{Z}^{2k} \to \mathbb{Z}$ be a positive definite quadratic form on $\mathbb{Z}^{2k}$ which takes integral values. The *theta function* associated to Q is defined as

$$\Theta_Q(\tau) = \sum_{v \in \mathbb{Z}^{2k}} \exp(2\pi i Q(v)\tau).$$

By a theorem of Hecke and Schoenberg [3, p. 32], the series Θ_Q defines a modular form of weight k with respect to some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and some character, both depending on Q.

The modular behavior of Θ_Q enables us to study quadratic forms by means of arithmetic properties of modular forms. In fact, the theta function previously introduced has a Fourier expansion of the form

$$\Theta_Q(\tau) = \sum_{n \geq 0} R_Q(n) \exp(2\pi i n\tau),$$

where $R_Q(n)$ is the number of elements $v \in \mathbb{Z}^{2k}$ such that $Q(v) = n$. From the theory of modular forms, we immediately obtain estimates of the growth of the representation numbers $R_Q(n)$ when n diverges.

Another application of theta series is their use as kernel functions to construct the so-called *theta lifts*. A theta lift of a weight k cusp form f with respect to $\mathrm{SL}_2(\mathbb{Z})$ may be defined as the Petersson inner product of f and some theta function $\Theta = \Theta(\tau, Z)$, namely

$$\Lambda_\Theta(f) \coloneqq \langle f, \Theta \rangle_{\mathrm{Petersson}} = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k f(\tau)\overline{\Theta(\tau, Z)}\, \frac{dx\, dy}{y^2},$$

where $\tau = x + iy$ and Z is a second auxiliary variable in (some generalization of) the upper-half plane $\mathbb{H}$. The theta function is constructed in such a way that the lift $\Lambda_\Theta(f)$ is an automorphic form with respect to the variable Z.

A theta lift can be interpreted as an explicit realization of the theta correspondence for a dual reductive pair (G, G') in the sense of Howe. An example of a dual reductive pair is given by a symplectic group G and an orthogonal group G', e.g., we may choose $G = \mathrm{SL}_2$ and $G' = \mathrm{O}(n, 2)$, where the latter is the standard indefinite orthogonal group.

A well-studied example of theta lifts is the Shimura–Shintani correspondence. It may be regarded as a bridge between modular forms of integral weight and modular forms of half-integral weight.

2. Outline

(1) Introduce Jacobi's theta series and the Eisenstein series of weight 1 and character χ_{-4}; see the beginning of [3, Section 3] and the remark on [3, p. 17]. Sketch the proof of the Theorem of Fermat as described in [3, pp. 26-27].
(2) Define theta series associated to positive definite quadratic forms and state the Theorem of Hecke–Schoenberg; see [3, Section 3.2].
(3) Introduce the basic idea of a theta lift, as explained in the introduction above. Illustrate the Shimura lift for half-integral cusp forms and describe it as a theta lift; see the introduction in [2].

References

[1] J.H. Bruinier, *Hilbert modular forms and their applications*, *The 1-2-3 Of Modular Forms*, pp. 105-179 (2008).
[2] S. Niwa, *Modular forms of half integral weight and the integral of certain theta-functions*, Nagoya Mathematical Journal Vol 56, pp. 147–161 (1975).
[3] D. Zagier, *Elliptic Modular Forms and Their Applications*, *The 1-2-3 Of Modular Forms*, pp. 2-103 (2008).

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