MAASS FORMS
INTRODUCTORY TALK ENTR-WORKSHOP

CONTACT PERSON: ANDREAS MONO

Abstract. Holomorphic modular forms are fruitful and therefore ubiquitous objects in analytic number theory. Real analytic modular forms are a natural generalization of holomorphic modular forms. Among the possible relaxations of holomorphicity and growth conditions, Maaß forms have proven to be of interest as well, in particular after Zweger's breakthrough on Ramanujan's mock theta functions. In this talk, we introduce the overall theory of harmonic Maaß forms, present some examples of them, and discuss an application of arithmetic/geometric nature. Proofs and further details can be found in [1].

1. Motivation

In [4], Maaß introduced new automorphic objects, to which we refer as *Maaß wave forms* nowadays. Roughly speaking, the definition of an automorphic form consists of

(1) a transformation law,
(2) an analytic condition,
(3) a growth condition.

In the archetypical example of holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$, conditions (2) and (3) are phrased as "holomorphicity" on $\mathbb{H}$ and at the cusp $i\infty$. Maaß' idea is to keep the transformation law of modular forms, but to relax the other to conditions. Instead of being holomorphic on $\mathbb{H}$, a Maaß form is required to be an eigenfunction of the *hyperbolic Laplace operator*

$$\Delta_k \coloneqq -v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)+ikv\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right),$$

where $\tau = u + iv \in \mathbb{H}$. Since Δ_k is an elliptic differential operator, its eigenfunctions are real analytic. The growth condition is also weakened to permit polynomial growth towards the cusps, to allow poles at the cusps, or even to relax to linear exponential growth towards them.

During the past decades, Fourier expansions and spectral theory of Maaß forms were established; see the work of Roelcke [5]. To mention one application of the theory, one can prove Duke's classical equidistribution theorem using the spectral theory of Maaß forms and a subconvex bound due to Conrey–Iwaniec [3].

The breakthrough of Maaß forms happened 20 years ago with Zwegers' thesis [6] on *mock theta functions*. These q-hypergeometric series, which were introduced by Ramanujan in his last letter to Hardy about 100 years ago, are not modular theirselves. Parallel to Zwegers' work, Bruinier and Funke [2] developed the theory of *harmonic Maaß forms*, i.e. Maaß forms with eigenvalue 0 under Δ_k. Zwegers constructed a *completion* g to each mock theta function f in the sense that $f+g$ is a harmonic Maaß

Date: July 27, 2022.

form. This clarifies the connection of Ramanujan's functions to a modular framework, which in turn founded new directions of modern research in combinatorics and (higher depth) mock modular forms.

Since then, many more applications of harmonic Maaß forms were found, since their Fourier coefficients often encode arithmetic or geometric information. A good illustration on the theory and some of its applications can be found in the book [1].

2. Outline

(1) Introduce harmonic Maaß forms as in [1, Section 4]. Remark that various growth conditions are used in the literature.
(2) Introduce the Fourier expansion of a harmonic Maaß form in general ([1, Lemma 4.3]).
(3) Define the *shadow operator* ξ_k and the *Bol operator* $\mathcal{D}^{1-k}$; see [1, Section 5]. Explain the action of both operators on the Fourier expansion. Emphasize that the result of each action is still modular.
(4) Define mock modular forms, shadows and mock theta functions as in [1, Section 5.4]. Discuss the two perspectives on harmonic Maaß forms briefly, as stated in the first remark on [1, p. 81].
(5) Present some examples of harmonic Maaß forms following [1, Sections 6.1, 6.3].
    (a) The completed Eisenstein series E_2^*
    (b) Zagier's non-holomorphic Eisenstein series of weight $\frac{3}{2}$.
    (c) Maaß–Poincaré series, which generalize holomorphic Poincaré series to negative weights.
    (d) Maaß–Eisenstein series.
(6) Present at least one application of your choice from [1, Section 16].

References

[1] K. Bringmann, A. Folsom, K. Ono, and L. Rolen, *Harmonic Maass forms and mock modular forms: theory and applications*, American Mathematical Society Colloquium Publications, vol. 64, American Mathematical Society, Providence, RI, 2017.
[2] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90.
[3] J. B. Conrey and H. Iwaniec, *The cubic moment of central values of automorphic L-functions*, Ann. of Math. (2) **151** (2000), no. 3, 1175–1216.
[4] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **121** (1949), 141–183 (German).
[5] W. Roelcke, *Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I, II*, Math. Ann. 167 (1966), 292–337; ibid. **168** (1966), 261–324 (German).
[6] S. Zwegers, *Mock theta functions*, Ph.D. Thesis, Ph.D. Thesis, Universiteit Utrecht, 2002.

Department of Mathematics and Computer Science, Division of Mathematics, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
Email address: amono@math.uni-koeln.de