# $L$-FUNCTIONS AND ELLIPTIC CURVES INTRODUCTORY TALK ENTR-WORKSHOP

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Abstract. $L$-functions are generalisations of the Riemann $\zeta$-function and may be attached to many geometric and arithmetic objects. They play a key role in the Langlands programme, which is probably the most ambitioned mathematical programmes of our time.

Elliptic curves are smooth projective curves of genus one with a fixed point. They are famously related to special values of $L$-functions by the Birch and Swinnerton-Dyer conjecture.

In this talk the overall theory of $L$-functions and elliptic curves is introduced and some examples and applications are discussed. Proofs and further details may be found in [1], [3], or [5].

## 1. Motivation

The starting point for the theory of $L$-functions is the well known *Riemann $\zeta$-function* which is defined as

$$\zeta(s) \coloneqq \sum_{n\geq 1} \frac{1}{n^s},$$

for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. It encodes the distribution of prime numbers and has remarkable properties. For instance, it has a meromorphic continuation to $\mathbb{C}$, $\Lambda(s) \coloneqq \pi^{-s/2}\Gamma(s/2)\zeta(s)$, an Euler product expansion and a functional equation of the following form

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \qquad \Lambda(1-s) = \Lambda(s).$$

For some arithmetic sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$, the associated $L$-function is a generalisation of $\zeta$ and may be given by a Dirichlet $L$-series

$$L(s) \coloneqq \sum_{n\geq 1} \frac{a_n}{n^s}$$

converging on a suitable right half plane. For instance, the arithmetic sequence may arise from the Fourier expansion of a modular form.
An interesting class of $L$-functions is that of *Dirichlet* $L$-functions, where $a_n := \chi(n)$ for some primitive Dirichlet character $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$. In this case we may write

$$L(s,\chi) \coloneqq \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-\chi(p)p^{-s}}.$$

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As for $\zeta$, these functions have a meromorphic continuation and satisfy a functional equation; see [2]. An interesting applications of Dirichlet $L$-functions is proving the existence of infinitely many primes in arithmetic progressions; see [7].

Another interesting class of $L$-functions was constructed by Hecke from characters of number fields, the so called *Größencharaktere*. This approach has been tackled adelically by Tate in his famous thesis in 1950, which is deemed to be the origin of the modern theory of automorphic representations. Those Größencharaktere play a central role in the *Langland's programme*, one of the most ambitious projects of modern mathematics.

In addition, the $L$-functions can be attached to certain algebraic curves, called *elliptic curves*. A (complex) elliptic curve is a quotient of the complex plane by a lattice. In particular, it is a torus of dimension one, an abelian group, and a compact Riemann surface. By means if the so called Weierstraß $\wp$-function, elliptic curves may be realised as cubic curves in $\mathbb{P}^2(\mathbb{C})$, defined by a *Weierstraß equation* of the form

$$y^2 = x^3 + Ax + B$$

for some constants $A, B$.
To every elliptic curve $E$ (defined over $\mathbb{Q}$) and every prime $p$, define the value $a_p(E)$ as the difference of $p$ and the number of solutions of the Weierstraß equation of $E$ over $\mathbb{F}_p$. These values may then be generalised by multiplicity to construct $a_n(E)$ for all $n \in \mathbb{N}$. The $L$-function attached to $E$ is the $L$-function arising from the sequence $(a_n(E))_{n\in\mathbb{N}}$. This construction plays a central role in Andrew Wiles' proof of Fermat's last Theorem.

## 2. Outline

(1) Introduce $L$-functions and elliptic curves as in [1, Chapter 19] and [3, Chapters 4 and 5].
(2) Explain which sort of singularities of elliptic curves exist; see [5, Chapter 8].
(3) Introduce the *Hasse-Weil $L$-function*; see [5] or [8, Appendix C.16]. Explain the good reduction and bad reduction at a prime number $p$, and state the local factors of the $L$-function.
(4) As an application, give a small introduction to the Birch and Swinnerton-Dyer Conjecture; see [1, Section 19.1].
(5) Present another example of an elliptic curve and its corresponding $L$-function.
(6) For the prime example of $L$-functions, sketch that it has meromorphic continuation and a functional equation by showing (see [5, Chapter 7, Section 5])

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = \int_0^\infty \frac{1}{2}\left(\theta(i\sigma) - 1\right)\sigma^{\frac{s}{2}-1}d\sigma,$$

splitting the integral adequately, and using modular properties of $\theta$ to derive a symmetric integral representation; cf. [4, Section 6.1].
(7) If time permits, present the gist of Dirichlet's proof for primes in arithmetic progressions; cf. [7, II.VI] might be helpful.

## References


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