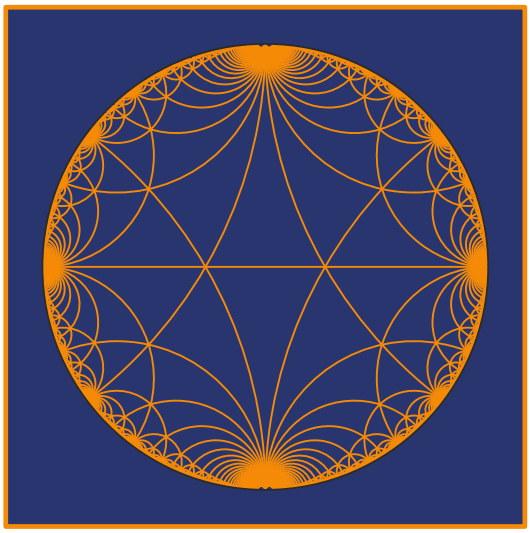

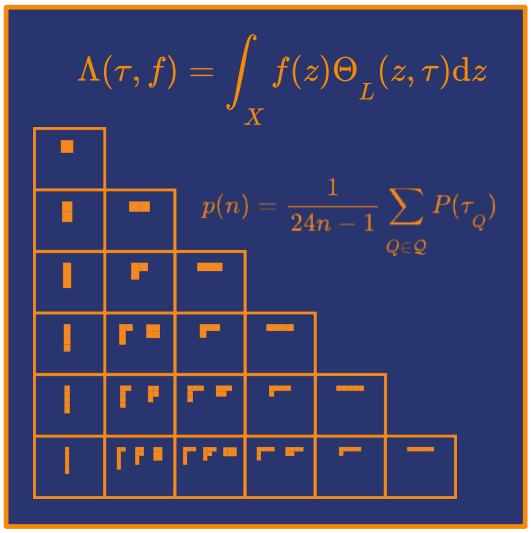

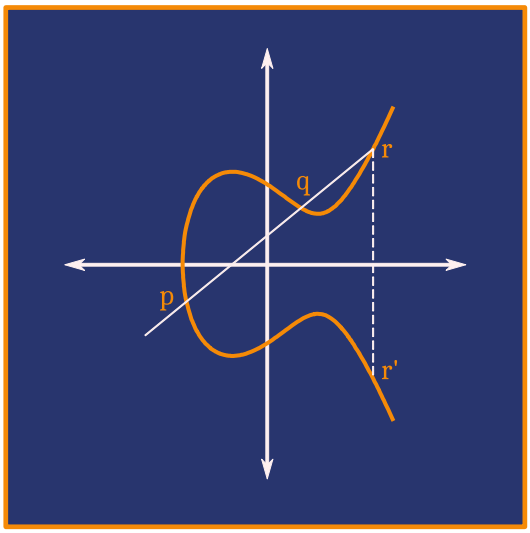

Early Number Theory Researchers Workshop

October 26 - 28, 2022

TU Darmstadt

Supported and funded by

GAUS

DFG Deutsche Forschungsgemeinschaft German Research Foundation

TECHNISCHE UNIVERSITÄT DARMSTADT

Contents

About

ENTR Workshop

This workshop is meant as an opportunity for community building and fostering collaborations between early career researchers in the field of number theory.

Format

The workshop will take three days. Each day, we focus on one of the topics which are listed below The day will start off with an introductory talk on the chosen topic, followed by short talks by young researchers and be topped off with a talk by an established researcher.

There will be plenty of time to discuss in between sessions and become acquainted over a casual cup of coffee for potential collaborations.

Topics

Real Analytic Modular Forms

Holomorphic modular forms are fruitful and therefore ubiquitous objects in analytic number theory. Real analytic modular forms are a natural generalization of holomorphic modular forms. Among the possible relaxations of holomorphicity and growth conditions, Maass forms have proven to be of interest as well, in particular after Zweger's breakthrough on Ramanujan's mock theta functions.

Theta Functions and Theta Lifts

Theta functions play an essential role in many areas of number theory. One exciting application is their use for the functional equation of the Riemann Zeta function and Zeta function of a number field. Further, theta functions are deeply intertwined with the theory of quadratic forms and their arithmetic properties.

A theta lift is an explicit application of the theta correspondence and can be seen as integrating automorphic forms against a theta function with similar automorphic properties. The result is again an automorphic form whose Fourier expansion encodes arithmetic information related to the input automorphic form.

L-Functions and Elliptic Curves

L-functions are generalizations of the Riemann Zeta-function and may be attached to many geometric and arithmetic objects. They play a key role in the Langlands program, which is one of the most ambitious mathematical programs of our time.

Elliptic curves are smooth projective curves of genus 1 with a fixed point. They are famously related to special values of L-functions by the Birch and Swinnerton-Dyer conjecture.

Organizing committee

- Cesana, Giulia (University of Cologne)
- Kleinemeier, Lars (Bielefeld University)
- Metzler, Ingmar (TU Darmstadt)
- Mono, Andreas (University of Cologne)
- Zuffetti, Riccardo (TU Darmstadt)

Timetable

IS: Invited Speaker, IT: Introductory Talk, CT: Contributed Talk, PD: Panel Discussion

Wednesday – L-Functions and elliptic curves

8:30–9:00	**Welcome**		
9:00–9:45	IT	**Paul Siemon** University of Bonn, Germany	Introductory Talk
9:45–10:15	CT	**Mirko Rösner** University of Heidelberg, Germany	The degree four L-function for $\mathrm{GSp}(4)$ and inner forms
10:15–10:45	**Coffee**		
10:45–11:45	IS	**Eugenia Rosu** Leiden University, Netherlands	Twists of elliptic curves with CM
11:45–12:15	CT	**Rashi Lunia** The Institute of Mathematical Sciences, Chennai, India	On Euler-Kronecker constants
12:15–13:45	**Lunch**		
13:45–14:15	CT	**Victoria Cantoral Farfán** University of Göttingen, Germany	Around the Sato–Tate conjecture
14:15–14:45	CT	**Michael Yiasemides** University of Nottingham, United Kingdom	Divisor Sums and Hankel Matrices
14:45–15:15	**Coffee**		
15:15-15:45	CT	**Manuel Müller** TU Darmstadt, Germany	The invariants of the Weil representation of $SL_2(\mathbb{Z})$

Thursday – Theta functions and lifts

9:15-10:00	IT	**Felix Pennig** TU Darmstadt, Germany	Introductory Talk
10:00-10:30	CT	**Paul Kiefer** TU Darmstadt, Germany	Lifting orthogonal modular forms
10:30–11:00	**Coffee**		
11:00–12:00	IS	**Yingkun Li** TU Darmstadt, Germany	Higher Green Function and Theta Lifts
12:00-12:30	CT	**Christina Röhrig** TU Darmstadt, Germany	Mock Maass Theta Functions
12:30–14:00	**Lunch**		
14:00–14:30	CT	**Xiaoyu Zhang** University of Duisburg-Essen, Germany	On the convergence of indefinite theta series
14:30–15:00	PD	**Jan Bruinier** TU Darmstadt, Germany	Career in Math
15:00–15:30	**Coffee**		
15:30–16:00	CT	**Brandon Williams** RWTH Aachen, Germany	Additive theta lifts that are Borcherds products
16:00-16:30	CT	**Mads Christensen** University College London, United Kingdom	Special Cycles in Arithmetic Hyperbolic Manifolds
16:30-17:00	CT	**Markus Schwagenscheidt** ETH Zurich, Switzerland	Theta functions and Borcherds products
17:15–18:30	**Walk to the Mathildenhöhe**		
19:00	**Conference Dinner**		

Friday – Real-Analytic Modular Forms

9:15-10:00	IT	**Johann Stumpenhusen** University of Cologne, Germany	Introductory Talk
10:00-10:30	CT	**Sunil Naik** The Institute of Mathematical Sciences, Chennai, India	Prime factorization of the Ramanujan tau function
10:30–11:00	**Coffee**		
11:00–12:00	IS	**Michael Mertens** University of Cologne, Germany	Weierstrass and Kleinian mock modular forms and related constructions
12:00-12:30	CT	**Qihang Sun** University of Illinois, Urbana-Champaign, USA	Uniform bounds on sum of Kloosterman sums and their application on partitions
12:30–14:00	**Lunch**		
14:00–14:30	CT	**Campbell Wheeler** Max Planck Institute for Mathematics, Bonn, Germany	Quantum modularity of some classical functions
14:30–15:00	PD	**Timo Richarz** TU Darmstadt, Germany	Career in Math
15:00–15:30	**Coffee**		
15:30–16:00	CT	**Maleeha Khawaja** University of Sheffield, United Kingdom	The Fermat equation over some totally real number fields
16:00-16:30	CT	**Rajat Gupta** Institute of Mathematics, Academia Sinica, Taiwan	Koshliakov zeta functions and modular relations
16:30-17:00	**Final Greetings**		

List of Abstracts – Talks

Wednesday – L-Functions and elliptic curves

The degree four L-function for $\mathrm{GSp}(4)$ and inner forms

Mirko Rösner **CT**

Mathematisches Institut, Universität Heidelberg, Germany

For the degree four L-function for automorphic representations of $\mathrm{GSp}(4)$, several constructions have been proposed. The most general is the one studied by Piatetskii-Shapiro and Soudry, but it depends on an arbitrary choice of a Bessel model.

In recent work with Weissauer and Danisman, we have shown that the non-archimedean local Euler factors are independent of this choice and are exactly the ones expected by the local Langlands correspondence in the sense of Gan and Takeda. The analogous results hold for the inner form $\mathrm{GSp}(1,1)$.

Twists of elliptic curves with CM

Eugenia Rosu **IS**

Leiden University, Leiden, Netherlands

We consider certain families of sextic twists of the elliptic curve $y^2 = x^3 + 1$ that are not defined over $\mathbb{Q}$, but over $\mathbb{Q}[\sqrt{-3}]$.

We compute a formula that relates the central value of their L-functions $L(E,1)$ to the square of a trace of a modular function evaluated at a CM point. When the value above is non-zero, we recover the order of the Tate-Shafarevich group, and we show that the value is indeed an integer square.

On Euler-Kronecker constants

Rashi Lunia **CT**

The Institute of Mathematical Sciences, Chennai, India

The Dedekind zeta function $\zeta_K(s)$ of a number field K has a simple pole at $s = 1$ with residue ρ_K. Its Laurent series expansion around $s = 1$ is given by

$$\zeta_K(s) = \frac{\rho_K}{s-1} + c_K + O(s-1).$$

Ihara defined the Euler-Kronecker constant of a number field K as $\gamma_K = \frac{c_K}{\rho_K}$. He conjectured that there exists positive constants $a_0, a_1 \leq 2$ such that for m sufficiently large and any $\epsilon > 0$, we have

$$(a_0 - \epsilon) \log m < \gamma_{\mathbb{Q}(\zeta_m)} < (a_1 + \epsilon) \log m.$$

This bound was shown to hold on average by Fouvry as we vary over a family of cyclotomic fields. In this talk, we elaborate on an ongoing work where we study the Euler-Kronecker constant of a more general family of number fields and compare our results with existing bounds.

Around the Sato–Tate conjecture

Victoria Cantoral Farfán **CT**

University of Göttingen, Germany

The famous Sato–Tate conjecture for elliptic curves defined over a number field (without complex multiplication) predicts the equidistribution of Frobenius traces with respect to the Haar measure of the corresponding Sato–Tate group under the trace map.

This conjecture has already been generalized for higher-dimensional abelian varieties, $K3$ surfaces, and pure motives of odd weight. During this talk, we will introduce the Sato–Tate conjecture and highlight its connections with the L-functions.

Divisor Sums and Hankel Matrices

Michael Yiasemides **CT**

University of Nottingham, United Kingdom

Sums involving divisor functions are of particular interest in number theory for several reasons. For example, the prominent open problem of higher moments of $\zeta(s)$ requires an understanding of certain divisor sums known as correlations.

Another area of interest is the mean, variance, and higher moments of divisor functions over short intervals. We can study such problems in another setting: Function fields. That is, the polynomial analogue of the integers. There are many analogies between integers and polynomials. For example, primes are analogous to irreducible polynomials, and we can formulate Riemann hypotheses in the polynomial setting.

In this talk we briefly demonstrate the importance of divisor sums, and briefly introduce the analogy between integers and polynomials. We then explain a new approach we have developed for solving divisor sums, which uses additive characters to translate the problem to one that is more linear algebraic in nature and involves Hankel matrices.

The invariants of the Weil representation of $SL_2(\mathbb{Z})$

Manuel Müller **CT**

TU Darmstadt, Germany

The transformation behaviour of the vector-valued theta function of a positive definite even lattice under the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ is described by the Weil representation. This representation plays an important role in the theory of automorphic forms. We show that its invariants are induced from 5 fundamental invariants.

Thursday – Theta functions and lifts

Lifting Orthogonal Modular Forms

Paul Kiefer **CT**

TU Darmstadt, Germany

We explain how to lift orthogonal modular forms of singular weight to obtain invariant vectors for the Weil representation.

Higher Green Function and Theta Lifts

Yingkun Li IS

TU Darmstadt, Germany

By the classical theory of complex multiplication, the modular j-function takes algebraic values at CM points. It is an interesting question to ask about the algebraic nature of other types of automorphic functions at CM points.

For the automorphic Green function at integral parameters, Gross and Zagier conjectured in the 1980s that their values at CM points are essentially logarithms of algebraic numbers.

In this talk, we will discuss recent progress toward this conjecture using theta lifts. This is partly joint with Jan Bruinier and Tonghai Yang.

Mock Maass Theta Functions

Christina Röhrig CT

TU Darmstadt, Germany

We interpret mock Maass theta functions as defined by Zwegers in the setting of theta lifts and discuss a generalization of this construction. This is joint work in progress with Yingkun Li.

On the convergence of indefinite theta series

Xiaoyu Zhang CT

University of Duisburg-Essen, Germany

There are various generalizations of Zwegers's indefinite theta series for quadratic spaces of signature (n, m). In this talk we consider one such generalization for $m = 2$ due to Alexandrov, Banerjee, Manschot and Pioline. We give a geometric interpretation of the kernel function and then present an elementary proof of the absolute convergence of such theta series.

Additive theta lifts that are Borcherds products

Brandon Williams CT

RWTH Aachen, Germany

We will show that the (singular) Gritsenko lift of every theta block of q-order one is a Borcherds product whose input is a linear combination of "auxiliary" theta blocks of weight zero. This contains the theta block conjecture of Gritsenko, Poor and Yuen as a special case. This is joint work with Haowu Wang.

Special Cycles in Arithmetic Hyperbolic Manifolds

Mads Christensen **CT**

University College London, United Kingdom

I will introduce certain submanifolds in hyperbolic space and explain how they give rise to cycles in quotients of hyperbolic space by arithmetic groups. The cohomology classes of these cycles turn out to be the coefficients of a modular form. If time permits, I will comment on how this can be proven using the Kudla-Millson theta series.

Theta functions and Borcherds products

Markus Schwagenscheidt **CT**

ETH Zurich, Switzerland

We show that all holomorphic weight 1 theta functions associated with positive definite integral binary quadratic forms have Borcherds-type infinite product expansions. In particular, the roots of these binary theta functions lie at CM points in the upper half-plane, and can be described very explicitly.

If time permits, we discuss some examples of higher weight theta functions that are Borcherds products, and some counter-examples, that is, theta functions which are not Borcherds products. This is joint work with Brandon Williams.

Friday – Real-Analytic Modular Forms

Prime factorization of the Ramanujan τ function

Sunil Naik **CT**

The Institute of Mathematical Sciences, Chennai, India

Investigation of the arithmetic properties of the Ramanujan τ function is a fascinating area in Number theory. A famous result of Hardy and Ramanujan states that the number of distinct prime factors of n has normal order $\log \log n$.

In this talk, I will present our recent work on the lower bound for the number of distinct prime factors of the Ramanujan τ function. As a consequence of this result, we will deduce a lower bound for the radicals of the Ramanujan τ function.

Higher Green Function and Theta Lifts

Michael Mertens IS

University of Cologne, Germany

When Bruinier and Funke introduced the notion of harmonic Maaß forms, they already showed that for any given cusp form g, there is a harmonic Maaß form F whose shadow is g. There are various theoretical ways to construct these forms, e.g. via Poincaré series or via holomorphic projection. An alternative way, which works specifically for newforms of weight 2 with rational coefficients, which is based on very classical concepts and works quite efficiently in practice, is through Weierstrass mock modular forms.

In my talk, I will discuss this construction and various generalizations of it, relaxing the requirement that the coefficients of the given newform be rational. This is joint work in progress with Claudia Alles-Neumann.

If time permits, I shall also discuss a generalization of the construction to higher weights, which is joint work in progress with Claudia Alles-Neumann, Jens Funke, and Eugenia Rosu.

Uniform bounds on sum of Kloosterman sums and their application on partitions

Qihang Sun CT

University of Illinois, Urbana-Champaign, USA

In studies of the partition function, Rademacher's exact formula is one step much better than the asymptotic from Hardy and Ramanujan. The exact formula is a sum of Kloosterman sums (with a Bessel function) and its convergence rate can be derived from estimates of that.

After the notion "rank" of a partition coming up from Dyson, its generating function connects properties of partitions and mock modular forms. Today we are going to talk about uniform estimates on sum of Kloosterman sums with a general family of multipliers, which improves the result by Goldfeld and Sarnak in 1983 and applies well on the exact formula of partitions with rank modulo 1 to 3.

Quantum modularity of some classical functions

Campbell Wheeler CT

Max Planck Institute for Mathematics, Bonn, Germany

I will briefly introduce holomorphic quantum modular forms and will outline the proof that Heine's q-hypergeometric functions satisfy this new form of modularity.

The Fermat equation over some totally real number fields

Maleeha Khawaja **CT**

University of Sheffield, United Kingdom

Let n be an integer greater than or equal to 3. Since the work of Wiles, the Fermat equation $x^n+y^n = z^n$ has been studied extensively over various number fields. Suppose (a, b, c) is a non-trivial solution to the Fermat equation over K for $n = p$.

Following the modular approach, we can associate the elliptic curve

$$E : y^2 = x(x - a^p)(x + b^p)$$

to this solution, which is defined over K.

In this talk, we will see how various other elliptic curves come into play when we fix the exponent of the Fermat equation and study it over certain totally real number fields.

Koshliakov zeta functions and modular relations

Rajat Gupta **CT**

Institute of Mathematics, Academia Sinica, Taiwan

Nikolai Sergeevich Koshliakov was an outstanding Russian mathematician who made phenomenal contributions to number theory and differential equations. In the aftermath of World War II, he was one among the many scientists who were arrested on fabricated charges and incarcerated. Under extreme hardships while still in prison, Koshliakov (under a different name 'N. S. Sergeev') wrote two manuscripts out of which one was lost.

Fortunately the second one was published in 1949 although, to the best of our knowledge, no one studied it until the last year when Prof. Atul Dixit and I started examining it in detail. This manuscript contains a complete theory of two interesting generalizations of the Riemann zeta function having their genesis in heat conduction and is truly a masterpiece!

In this talk, we will discuss some of the contents of this manuscript and then proceed to give some new results (modular relations) that we have obtained in this theory. This is joint work with Prof. Atul Dixit.

List of Participants

Surname, First name	Affiliation
Bieker, Patrick	TU Darmstadt
Burmester, Annika	University of Hamburg
Cantoral Farfán, Victoria	University of Göttingen
Cesana, Giulia	University of Cologne
Christensen, Mads	University College London
Dang, Thuong	Heinrich Heine University Düsseldorf
Eichler, Jaro	Goethe University Frankfurt
Gosh, Aditya	University of Oxford
Gupta, Rajat	Institute of Mathematics, Academia Sinica
Göken, Jana	University of Bremen
Khawaja, Maleeha	University of Sheffield
Kiefer, Paul	TU Darmstadt
Kleinemeier, Lars	Bielefeld University
Li, Yingkun	TU Darmstadt
Lunia, Rashi	Institute of Mathematical Sciences, Chennai
Mertens, Michael	University of Cologne
Metzler, Ingmar	TU Darmstadt
Mono, Andreas	University of Cologne
Müller, Manuel	TU Darmstadt
Naik, Sunil	Institute of Mathematical Sciences, Chennai
Pennig, Felix	TU Darmstadt
Röhrig, Christina	TU Darmstadt
Rösner, Mirko	University of Heidelberg
Rosu, Eugenia	Leiden University
Schwagenscheidt, Markus	ETH Zurich
Sevenig, Dennis	RWTH Aachen
Siemon, Paul	University of Bonn
Stumpenhusen, Johann	University of Cologne
Sun, Qihang	University of Illinois, Urbana-Champaign
van Ittersum, Jan-Willem	Max Planck Institute for Mathematics
Wheeler, Campbell	Max Planck Institute for Mathematics
Williams, Brandon	RWTH Aachen
Yiasemides, Michael	University of Nottingham
Zhang, Mingkuan	TU Darmstadt
Zhang, Pengcheng	Max Planck Institute for Mathematics
Zhang, Xiaoyu	University of Duisburg-Essen
Zuffetti, Riccardo	TU Darmstadt
zur Verth, Svenja	EPFL

Useful Information

Directions, Location and Lecture hall

The talk will take place in the "Universität- und Landesbibliothek" S1|20, lecture hall R.01. The coffee breaks take place in the same building. The lecture hall is located in the basement. The address of the building is "Magdalenenstraße 8, 64289 Darmstadt, Germany".

From Darmstadt Hauptbahnhof, one can take the bus lines F (Oberwaldhaus), H (Kranichstein Kesselhutweg), or K (Alfred-Messel-Weg). Get off at the stop "Alexanderstraße/TU."

Walk to the Mathildenhöhe and Workshop dinner

On Thursday afternoon, we will walk together to the Mathildenhöhe from the lecture hall at 5:15 pm. It takes approximately 15 minutes, so the group will arrive at the Mathildenhöhe at 5:30 pm if you want to join us there.

The conference dinner takes place at Sardegna after the walk to the Mathildenhöe on Thursday at 7 pm. Please be aware that every participant has to pay for the dinner themself and only the meal of the invited main speakers will be covered.

Food & Beverage

There is a number of restaurants and bars to choose from which are close to the university.
Restaurants:

- Sushi: The Ichido is an all you can eat restaurant while Yang Ji is a small restaurant for ordering (you cannot reserve a table here, but they offer takeaway).
- Italian: The Sardegna is a classic Italian restaurant close to the park.
- German: Die Sitte is a classic old German restaurant.
- Pizza: L'Osteria offers huge pizzas with many toppings and you may watch your pizza be garnished and baked.
- Eritraen: The Baobab offers authentic Eritrean food, including self-made slightly sour flat bread.
- Kebap: Of course, there are many diners or takeaways offering Döner or French tacos, which are popular in Germany.

- Breakfast: The Cafe Chaos offers breakfast all day; in case you had a rough night…
- High cuisine: If you happen to have the extra buck and plan on enjoying excellent cuisine, you might want to reserve a table at the OX. It has been awarded a Michelin star recently.

Bars and pubs:

- The Hobbit is a small, dark pub very close to the university which offers Laternchen, a local special drink, and some processed food.
- The Hotzenplotz is a similar pub nearby – a rival.
- The 3 Klang's menu changes every week and the food has a surprising quality for a bar. You can also sit outside here and drink a coffee in the afternoon.
- The Sausalitos is a big franchise bar offering drinks and mexican food.
- The Corroboree is an Australian style franchise bar offering drinks and kangeroo and crocodile meat.
- The An Sibin is a dark Irish pub offering karaoke and pub quizzes.
- The Green sheep is an Irish pub popular amongst students.

Covid Information

There are several restriction at the TU Darmstadt which can be found in detail here.

In short: Wearing a medical mask is still required in most situations. There are some exceptions, listed on the Covid information website of TU Darmstadt, where one is allowed to take off the mask (e.g. for drinking and eating). We will brief everyone about the exact restriction at the beginning of the workshop on Thursday.

Sponsors

The ENTR workshop is part of the GAUS CRC 326, funded by the German Research Forderation (DFG).