Holomorphic extensions related to a basic set of the 2nd largest Fischer group

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3-transposition groups

Definition 1

(G, I) : 3-transposition group

$\iff$ G : group, I : a set of involutions s.t. $G = \langle I \rangle$, $I^G = I$ and $|ab| \leqslant 3$ for $\forall a, b \in I$.

Theorem 1 (Fischer'71, Cuypers-Hall'95)

The list of almost simple finite 3-transposition groups is as follows.

1. $G = \mathrm{Sym}_n$, $I = \{(i\ j) \mid 1 \leqslant i < j \leqslant n\}$
2. $G = \mathrm{O}^{\pm}_{2n}(2)$, I = transvections
3. $G = \mathrm{Sp}_{2n}(2)$, I = transvections
4. $G = \mathrm{O}^{\pm}_{n}(3)$, I = reflections
5. $G = \mathrm{SU}_n(2)$, I = transvections
6. $G = \mathrm{P}\Omega^{+}_{8}(2 \text{ or } 3){:}\mathrm{Sym}_3$, $\mathrm{Fi}_{22,23,24}$, I : unique

Basic sets

Definition 2

Let (G, I) be a finite 3-transposition group and P a Sylow 2-subgroup.

The intersection $P \cap I$ is called a basic set of G and the size $|P \cap I|$ is called the width of G.

- $a, b \in P \cap I \implies |ab| = 2 \implies \langle P \cap I \rangle$: elementary abelian 2-group
- $P \cap I$: a maximal set of mutually commutative elements in I
- $\forall g \in G, (P \cap I)^g = P^g \cap I^g = P^g \cap I \implies |P \cap I|$: invariant of G (= width)

	Fi_{22}	Fi_{23}	Fi_{24}
Width	22	23	24
$\langle P \cap I \rangle$	2^{10}	2^{11}	2^{12}
Normalizer	$2^{10}.M_{22}$	$2^{11}.M_{23}$	$2^{12}.M_{24}$

CFSG

Theorem 2

The complete list of finite simple group is given by:

(0) $\mathbb{Z}/p\mathbb{Z}$ (p : prime)

(1) $\mathrm{Alt}_{n\geqslant 5}$

(2) Groups of Lie type (i.e. matrix groups over finite fields)

(3) 26 sporadics ($\mathrm{M}_{11,12,22,23,24}$, $\mathrm{Co}_{1,2,3}$, $\mathrm{Fi}_{22,23,24}$, $\mathbb{B}$, $\mathbb{M}, \ldots$)

Special symmetries in 24 dimension (cf. [FLM'88]):

$$
\begin{array}{ccccc}
\mathbb{F}_2^{24} & & \mathbb{Z}^{24} & & c = 24 \text{ VOA} \\
\mathcal{G} & \rightsquigarrow & \Lambda & \rightsquigarrow & V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T+} \\
\mathrm{M}_{24} & & 2.\mathrm{Co}_1 & & \mathbb{M}
\end{array}
$$

Note that $C_{\mathbb{M}}(2B) = 2_+^{1+24}.\mathrm{Co}_1$ and $\mathrm{Aut}(V_\Lambda^+) = 2^{24}.\mathrm{Co}_1 \cong (\Lambda/2\Lambda).\mathrm{Co}_1$

Sunshine construction?

Historically,

$$\begin{array}{ccccc} \mathrm{Fi}_{24} & \rightsquigarrow & \mathbb{B} & \rightsquigarrow & \mathbb{M} \\ \text{3-trans.gp.} & & \{3,4\}\text{-trans.gp.} & & \text{6-trans.gp.} \end{array}$$

Aim

$$\begin{array}{ccc} & \mathbb{M} & \\ \diagup & & \diagdown \\ 2.\mathbb{B} & & 3.\mathrm{Fi}_{24} \\ \diagdown & & \diagup \\ & S_3 \times \mathrm{Fi}_{23} & \\ & | & \\ & S_3 \times 2^{11} & \end{array} \Longrightarrow \begin{array}{ccc} & V^{\natural} & \\ \diagup & & \diagdown \\ L(1/2,0) \otimes VB^{\natural,\bar{0}} & & W_3(4/5) \otimes VF^{\natural} \\ \diagdown & & \diagup \\ & U_{3A} \otimes \mathrm{Com}_{V^{\natural}} U_{3A} & \\ & |\,? & \\ & X^{[23]} & \end{array}$$

Note that $C_{\mathbb{M}}(2A) = 2.\mathbb{B}$ and $N_{\mathbb{M}}(3A) = 3.\mathrm{Fi}_{24}$ (cf. $S_3 = 3{:}2$)

Ising vectors

Let V be a VOA of OZ-type, i.e. $V = \oplus_{n \geqslant 0} V_n$, $V_0 = \mathbb{R}\mathbb{1}$, $V_1 = 0$.

Then V_2 forms a commutative algebra with invariant bilinear form

$$ab := a_{(1)}b, \quad (a|b)\mathbb{1} = a_{(3)}b \quad \text{for} \quad a, b \in V_2.$$

This algebra is called the Griess algebra of V.

Lemma 3 (Miyamoto'96)

$e \in V_2 : c = c_e$ Virasoro vector $\iff$ $ee = 2e$ and $c_e = 2(e|e)$

Definition 3

$e \in V_2$: Ising vector of σ-type

$\iff$ $e : c = 1/2$ Virasoro vector s.t. $\langle e \rangle \cong L(1/2, 0)$ (simple subVOA)

There is no $\langle e \rangle$-submodule isomorphic to $L(1/2, 1/16)$ in V

Miyamoto involutions

Theorem 4 (Miyamoto'96)

If $e \in V$ is an Ising vector then $\tau_e := (-1)^{16o(e)} \in \mathrm{Aut}(V)$.

If $\tau_e = \mathrm{id}_V$ then $\sigma_e := (-1)^{2o(e)} \in \mathrm{Aut}(V)$.

Theorem 5 (Conway'85, Miyamoto'96, Höhn'10)

$$\begin{array}{ccc} \text{Ising vectors of } V^\natural & \overset{1:1}{\longleftrightarrow} & \text{2A-involutions of } \mathbb{M} \\ \Uplus & & \Uplus \\ e & & \tau_e \end{array}$$

By this correspondence, we can analyze 2A-involutions of $\mathbb{M}$ by considering corresponding Ising vectors of $V^\natural$.

We can generalize the above correspondence for the other groups.

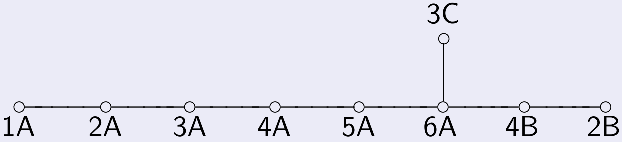
(cf. Lam-Y.'16 arXiv:1604.04989)

Dihedral subalgebras

Theorem 6 (Sakuma'07)

$e, f \in V_{\mathbb{R}}$: Ising vectors $\implies$ $|\tau_e\tau_f| \leqslant 6$ (6-transposition property)

More precisely, there are 9 possible types of $\langle e, f\rangle$:



$\langle e, f\rangle$	1A	2A	3A	4A	5A	6A	4B	2B	3C
$2^{10}(e \mid f)$	2^8	2^5	13	2^3	6	5	2^2	0	2^2
$\dim \langle e, f\rangle_2$	1	3	4	5	6	8	5	2	3

We will call $U_{nX} = \langle e, f\rangle$ the dihedral subalgebra of type nX.

Construction of 3-transposition groups I

V : VOA of OZ-type

D_V : the set of Ising vectors of σ-type in V

Set $\sigma(D_V) = \{\sigma_e \mid e \in D_V\}$ and $G_V = \langle D_V \rangle$.

If $e, f \in D_V$ then $\langle e, f \rangle$ is either 1A, 2A or 2B-type.

Theorem 7 (Miyamoto'96, Matsuo'05, Cuipo-Lam-Y.'18)

(1) $(G_V, \sigma(D_V))$: 3-transposition group of symplectic type (i.e. $G_V/O_2(G_V) \leqslant \mathrm{Sp}_{2n}(2)$)

(2) If $V = \langle D_V \rangle$ then (V, G_V) are classified (V is a subVOA of $V_{\sqrt{2}R}^{+}$)

$\langle e, f \rangle$: 2A-type $\implies$ $[\tau_e, \tau_f] = 1$ on V and $|\sigma_e \sigma_f| = 3$ on $V^{\langle \tau_e, \tau_f \rangle}$

$\langle e, f \rangle$: 2B-type $\implies$ $[\tau_e, \tau_f] = 1$ on V and $|\sigma_e \sigma_f| \leqslant 2$ on $V^{\langle \tau_e, \tau_f \rangle}$

In order to realize Fischer groups, we need to use τ-involutions.

Construction of 3-transposition groups II

V : VOA of OZ-type

E_V : the set of Ising vectors of V (including both σ and τ-types)

Fix $a,\ b \in E_V$ s.t. $\langle a, b\rangle$: 3A-type $\implies$ $\langle \tau_a, \tau_b\rangle \cong S_3$

Set $I_{a,b} := \{x \in E_V \mid (a\,|\,x) = (b\,|\,y) = 2^{-5}\}$ (i.e. $\langle a, x\rangle \cong \langle b, x\rangle \cong$ 2A-alg.)

Theorem 8 (Lam-Y.'16)

(1) $x \in I_{a,b} \implies [\tau_x, \tau_a] = [\tau_x, \tau_b] = 1$

(2) $x,\ y \in I_{a,b} \implies \langle x, y\rangle$: 1A, 2A or 3A-types

(3) $x,\ y \in I_{a,b} \implies |\tau_x\tau_y| \leqslant 3$ in $\mathrm{Aut}(V)$

(4) $G_V := \langle \tau_x \mid x \in I_{a,b}\rangle \implies G_V$: 3-transposition group in $C_{\mathrm{Aut}(V)}\langle \tau_a, \tau_b\rangle$

If we apply the theorem above to $V^\natural$ then we obtain $G_{V^\natural} = \mathrm{Fi}_{23} = C_{\mathbb{M}}(S_3)$.

Note that $S_3 \times \mathrm{Fi}_{23} < \mathbb{M}$ whereas $3.\mathrm{Fi}_{24} < \mathbb{M}$ but $\mathrm{Fi}_{24} \not< \mathbb{M}$.

Inductive structures

(G, I) : 3-transposition group

Pick $a \in I$ and set $G^{[1]} := G_a = \langle x \in I \mid ax = xa \rangle / \langle a \rangle$ $(G_a \leqslant C_G(a)/\langle a \rangle)$

Similarly we define $G^{[2]} := G_{a,b} = (G_a)_b$, $G^{[3]} := G_{a,b,c} = (G_{a,b})_c, \ldots$

Example

$$G = \mathrm{Fi}_{24} \implies G^{[1]} = \mathrm{Fi}_{23},\ \ G^{[2]} = \mathrm{Fi}_{22},\ \ G^{[3]} = \mathrm{Fi}_{21} = \mathrm{PSU}_6(2)$$

In the above process, $a, b, c, \ldots$: mutually commutative elements in I

Maximal collection : a basic set of (G, I)

Inductive subalgebras

$G_V = \langle \tau_x \mid x \in I_{a,b} \rangle$: 3-transposition group

Let n be the width of G_V. We will recursively define the inductive subalgebra

$$X^{[n]} := \langle a, b, x^1, \dots, x^n \rangle$$

as follows. First, we set $X^{[0]} := \langle a, b \rangle$ and suppose we have defined $X^{[i]} := \langle a, b, x^1, \dots, x^i \rangle$. Then we choose $x^{i+1} \in I_{a,b}$ s.t.

$$x^{i+1} \notin X^{[i]} \text{ and } (x^{i+1} \mid x^j) = 2^{-5},\ 1 \leqslant j \leqslant i$$

and define $X^{[i+1]} := \langle X^{[i]}, x^{i+1} \rangle = \langle a, b, x^1, \dots, x^i, x^{i+1} \rangle$ as long as possible.

Then $\{\tau_{x^i} \mid 1 \leqslant i \leqslant n\}$ gives a basic set of G_V if G_V is connected.

Inductive structures in VOA side

Set $D^{[0]} := \{\tau_x \mid x \in I_{a,b}\}$ and $D^{[i]} := \{\tau_y \in D^{[i-1]} \mid \tau_y \tau_{x^i} = \tau_{x^i} \tau_y\}$.

$$
\begin{array}{ccccccccc}
X^{[0]} & \subset & X^{[1]} & \subset & X^{[2]} & \subset & \cdots & \subset & X^{[n]} \\
\rightsquigarrow\!\downarrow & & \rightsquigarrow\!\downarrow & & \rightsquigarrow\!\downarrow & & & & \rightsquigarrow\!\downarrow \\
\mathrm{Com}_V X^{[0]} & \supset & \mathrm{Com}_V X^{[1]} & \supset & \mathrm{Com}_V X^{[2]} & \supset & \cdots & \supset & \mathrm{Com}_V X^{[n]} \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & & & \circlearrowleft \\
G^{[0]} & & G^{[1]} & & G^{[2]} & & \cdots & & G^{[n]} \\
\twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
\langle D^{[0]} \rangle & \supset & \langle D^{[1]} \rangle & \supset & \langle D^{[2]} \rangle & \supset & \cdots & \supset & \langle D^{[n]} \rangle
\end{array}
$$

Theorem 9 (Lam-Y.'16)

The Griess algebra of $X^{[n]}$ is uniquely determined and

$$L(c_3, 0) \otimes L(c_4, 0) \otimes \cdots \otimes L(c_{n+4}, 0) \subset X^{[n]} \quad \text{(full)}$$

where $c_i = 1 - 6/(i+2)(i+3)$ (unitary series).

Main example

When $V = V^\natural$, we obtain $G^{[0]} = \mathrm{Fi}_{23}$, $G^{[1]} = \mathrm{Fi}_{22}$ and $G^{[2]} = \mathrm{Fi}_{21} = \mathrm{PSU}_6(2)$.

$$
\begin{array}{ccccccc}
X^{[0]} & \subset & X^{[1]} & \subset & X^{[2]} & \subset & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Com}_{V^\natural} X^{[0]} & \supset & \mathrm{Com}_{V^\natural} X^{[1]} & \supset & \mathrm{Com}_{V^\natural} X^{[2]} & \supset & \cdots \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \\
G^{[0]} = \mathrm{Fi}_{23} & & G^{[1]} = \mathrm{Fi}_{22} & & G^{[2]} = \mathrm{PSU}_6(2) & & \cdots \\
\twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & \\
\langle D^{[0]} \rangle & \supset & \langle D^{[1]} \rangle & \supset & \langle D^{[2]} \rangle & \supset & \cdots
\end{array}
$$

Note that $X^{[n]} = \langle a, b, x^1, \ldots, x^n \rangle$ where $\{\tau_{x^1}, \ldots, \tau_{x^n}\}$ is a set of mutually commutative transpositions in $D^{[0]}$.

For $G^{[0]} = \mathrm{Fi}_{23}$, its width is 23 so that there exist 23 Ising vectors $x^1, \ldots, x^{23} \in V^\natural$ such that $X^{[23]} = \langle a, b, x^1, \ldots, x^{23} \rangle \subset V^\natural$ (cf. [Conway-Miyamoto]).

Observation

The central charge of $X^{[n]}$ is $c_3 + c_4 + \cdots + c_{n+4} = \dfrac{(n+2)(5n+29)}{5(n+7)}$.

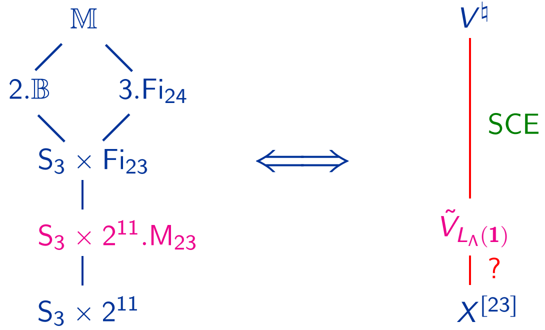
If $n = 23$: the central charge of $X^{[23]}$ is 24

On the other hand, $\mathrm{S}_3 \times \mathrm{Fi}_{23} < \mathbb{M} \implies X^{[23]} \subset V^\natural$ [Conway-Miyamoto]

$$L(c_3, 0) \otimes \cdots \otimes L(c_{27}, 0) \subset X^{[23]} \subset V^\natural$$

finite extension

(cf. T.Creutzig @Dubrovnik 2017)



Construction of $2+23$ involutions

Let $n \equiv 8 \mod 16$.

$F = [\epsilon_1, \ldots, \epsilon_n]_{\mathbb{Z}} = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$, $(\epsilon_i \,|\, \epsilon_j) = 2\delta_{i,j}$ (2-frame)

$F^* = \frac{1}{2}F$: dual lattice of F

$\pi : F^* \twoheadrightarrow F^*/F \cong \mathbb{F}_2^n \supset C$: code $\rightsquigarrow$ $L_A(C) := \pi^{-1}(C)$: lattice

$L_A(C)$: even, unimodular iff C : doubly-even, self-dual

$F^* \ni \mathbf{1} = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$ $\leftarrow\!\!-\!-\!-\!\!\rightarrow$ $(\bar{1}\bar{1}\cdots\bar{1}) \in \mathbb{F}_2^n$ (all-one vector)

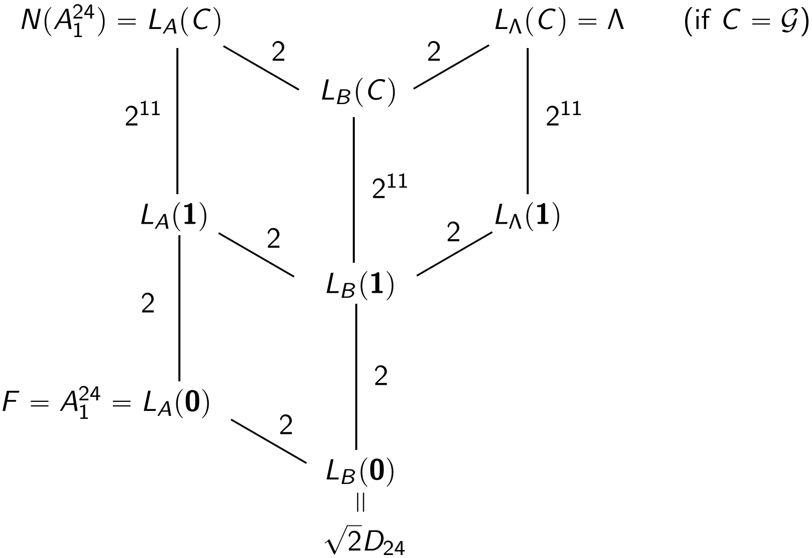
$\nu_C : L_A(C) \ni x \longmapsto \overline{(x \,|\, \mathbf{1})} \in \mathbb{F}_2$ $\rightsquigarrow$ $L_B(C) := \nu_C^{-1}(\bar{0})$: sublattice

For $(\bar{1}\bar{1}\cdots\bar{1}) \in C < \mathbb{F}_2^n$, set

$$L_\Lambda(C) := L_B(C) \sqcup (\underbrace{L_B(C) + \epsilon_1}_{=\nu_C^{-1}(\bar{1})} + \tfrac{1}{2}\mathbf{1})$$

$n = 24$, $C = \mathcal{G} < \mathbb{F}_2^{24}$ (Golay code) $\Longrightarrow$ $L_A(\mathcal{G}) = N(A_1^{24})$, $L_\Lambda(\mathcal{G}) = \Lambda_{24} = \Lambda$

Construction of $2 + 23$ involutions



Construction of $2+23$ involutions

Set $\alpha_0 = \frac{1}{2}\mathbf{1} - \epsilon_1$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(1 \leqslant i \leqslant 23)$, $\alpha_{24} = \epsilon_{23} + \epsilon_{24}$

$$[\alpha_0, \alpha_1, \dots, \alpha_{23}]_{\mathbb{Z}} \cong \sqrt{2}A_{24}$$

$$L_B(\mathbf{0}) = [\alpha_1, \dots, \alpha_{24}]_{\mathbb{Z}} \cong \sqrt{2}D_{24}$$

$$L_B(\mathbf{1}) = [\alpha_2, \dots, \alpha_{24}, \mathbf{1}]_{\mathbb{Z}} \cong \sqrt{2}N(D_{24}) \text{ (totally even \& 2-elementary)}$$

$$L_\Lambda(\mathbf{1}) = L_B(\mathbf{1}) \sqcup (L_B(\mathbf{1}) + \alpha_0) = [\alpha_2, \dots, \alpha_{24}, \alpha_0 + \mathbf{1}]_{\mathbb{Z}}$$

$$w^{\pm}(\alpha_i) = \frac{1}{16}\alpha_{i(-1)}{}^2\mathbb{1} \pm \frac{1}{4}(e^{\alpha_i} + e^{-\alpha_i}) \in V^+_{L_\Lambda(\mathbf{1})} \text{ : Ising vector [DMZ]}$$

Set $a := w^-(\alpha_0)$, $x^i := w^-(\alpha_i)$ $(1 \leqslant i \leqslant 23)$

Lemma 10

$(a \,|\, x^i) = (x^j \,|\, x^k) = 2^{-5}$ for $1 \leqslant i \leqslant 23$ and $1 \leqslant j < k \leqslant 23$.

Construction of 2 + 23 involutions

Theorem 11 (Abe-Dong-Li'05, van Ekeren-Möller-Scheithauer'17)

V_L^+ has group-like fusion $\iff$ L : 2-elementary (i.e. $2L^* < L$)

Both $V_{L_B(\mathbf{1})}^+$ and $V_{L_\Lambda(\mathbf{1})}^+$ have group-like fusion: $\mathscr{F}(V_{L_B(\mathbf{1})}^+)^\circ \cong 2^{26}$ and $\mathscr{F}(V_{L_\Lambda(\mathbf{1})}^+)^\circ \cong 2^{24}$

χ : trivial character of $L_B(\mathbf{1})/2L_B(\mathbf{1})$ $\qquad$ $V_{L_B(\mathbf{1})}^\chi$: $\mathbb{Z}_2$-twisted $V_{L_B(\mathbf{1})}$-module affording χ

$$\tilde{V}_{L_\Lambda(\mathbf{1})} = \underbrace{V_{L_B(\mathbf{1})}^+ \oplus V_{L_B(\mathbf{1})+\alpha_0}^+}_{\subset V_\Lambda^+} \oplus \underbrace{V_{L_B(\mathbf{1})}^{\chi+} \oplus V_{L_B(\mathbf{1})+\alpha_0}^{\chi+}}_{\subset V_\Lambda^{T+}} \subset V^\natural$$

where $V_{L_B(\mathbf{1})+\alpha_0}^{\chi+} := V_{L_B(\mathbf{1})+\alpha_0}^+ \boxtimes_{V_{L_B(\mathbf{1})}^+} V_{L_B(\mathbf{1})}^{\chi+}$.

Proposition 12 (FLM'88)

$\exists \rho \in \mathrm{Aut}(V_{L_B(\mathbf{1})}^+)$ s.t. $\rho^2 = 1$ and $\rho : V_{L_B(\mathbf{1})+\alpha_0}^+ \longleftrightarrow V_{L_B(\mathbf{1})}^{\chi+}$

$\Longrightarrow$ ρ can be extended to an automorphism of $\tilde{V}_{L_\Lambda(\mathbf{1})}$ of order 2.

Construction of $2 + 23$ involutions

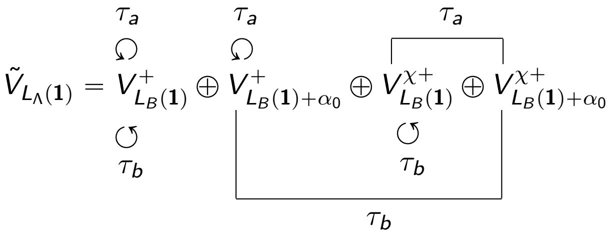
Set $b := \rho a \in \tilde{V}_{L_\Lambda(\mathbf{1})}$

Lemma 13

(1) $(a \,|\, b) = 13/2^{10}$ and $\langle a, b\rangle$: 3A-alg.

(2) $(b \,|\, x^i) = 2^{-5}$ for $1 \leqslant i \leqslant 23$.

(3) $X^{[23]} = \langle a, b, x^1, \ldots, x^{23}\rangle \subset \tilde{V}_{L_\Lambda(\mathbf{1})} \subset V^\natural$



Mathieu$_{23}$

$\Lambda = L_\Lambda(\mathcal{G}) \implies V^\natural \supset \tilde{V}_{L_\Lambda(\mathbf{1})}$: $\mathcal{G}/\mathbf{1}$-graded SCE where $\mathcal{G}/\mathbf{1} \cong 2^{11}$ as an abstract group

Proposition 14

Set $H := \langle \tau_{x^i} \mid 1 \leqslant i \leqslant 23 \rangle$. Then $H \cong 2^{11}$ and $(V^\natural)^H = \tilde{V}_{L_\Lambda(\mathbf{1})}$.

Note that $\langle x^1, \ldots, x^{23} \rangle \cong M_{A_{23}} \subset \tilde{V}_{L_\Lambda(\mathbf{1})} = (V^\natural)^H$ (cf. [Lam-Cuipo-Y'19])

$\implies G = \langle \sigma_{x^i} \mid 1 \leqslant i \leqslant 23 \rangle \cong W(A_{23}) = \mathrm{S}_{24} \circlearrowleft \tilde{V}_{L_\Lambda(\mathbf{1})}$

Lemma 15 (Shimakura'06)

Let $g \in G = \langle \sigma_{x^i} \mid 1 \leqslant i \leqslant 23 \rangle \cong \mathrm{S}_{24}$.

Then $\exists \hat{g} \in \mathrm{Aut}(V^\natural) = \mathbb{M}$ s.t. $\hat{g}|_{\tilde{V}_{L_\Lambda(\mathbf{1})}} = g \iff g \in \mathrm{M}_{23} = \mathrm{Aut}(\mathcal{G}/\mathbf{1})$

Theorem 16 (Creutzig-Lam-Y.)

$\mathrm{Stab}_{\mathbb{M}}(\tilde{V}_{L_\Lambda(\mathbf{1})}) \supset 2^{11}.\mathrm{M}_{23} = (\mathcal{G}/\mathbf{1})^\vee.\mathrm{Aut}(\mathcal{G}/\mathbf{1})$

Extensions

Consider extensions of $\tilde{V}_{L_\Lambda(\mathbf{1})} = V^+_{L_\Lambda(\mathbf{1})} \oplus V^{\chi+}_{L_\Lambda(\mathbf{1})}$ where

$$V^+_{L_\Lambda(\mathbf{1})} = V^+_{L_B(\mathbf{1})} \oplus V^+_{L_B(\mathbf{1})+\alpha_0} \subset V^+_\Lambda \quad \text{and} \quad V^{\chi+}_{L_\Lambda(\mathbf{1})} = V^{\chi+}_{L_B(\mathbf{1})} \oplus V^{\chi+}_{L_B(\mathbf{1})+\alpha_0} \subset V^{T+}_\Lambda$$

Note that $V^+_{L_\Lambda(\mathbf{1})}$ has group-like fusion such that $\mathscr{F}(V^+_{L_\Lambda(\mathbf{1})})^\circ \cong 2^{24}$ by Theorem 11.

Theorem 17 (van Ekeren-Möller-Scheithauer'17, Creutzig-Kanade-Linshaw'19)

Suppose Rep(V) is unitary and modular. Then a simple current extension of V corresponds to a choice of totally isotropic subgroup of $\mathscr{F}(V)^\circ$.

Proposition 18

W : an extension of $\tilde{V}_{L_\Lambda(\mathbf{1})} \implies \exists C \subset \mathbb{F}_2^{24}$: doubly-even code s.t. $W \cong \tilde{V}_{L_\Lambda(C)}$

In particular, a holomorphic extension of $\tilde{V}_{L_\Lambda(\mathbf{1})}$ is given by $\tilde{V}_{L_\Lambda(C)}$ where C is a doubly-even self dual code of $\mathbb{F}_2^{24}$.

Main result

Theorem 19 (FLM'88, DGM'96, Creutzig-Lam-Y.)

$X^{[23]} \subset \tilde{V}_{L_\Lambda(\mathbf{1})} \subset V$: $c = 24$ holomorphic VOA

$\iff V = \tilde{V}_{L_\Lambda(C)}$ with C : doubly-even self-dual binary code of length 24

In particular, there exist 8 holomorphic conformal extensions of $X^{[23]}$.

C	$d_{10}e_7^2$	d_{24}	d_{12}^2	d_6^4	e_8^3, $d_{16}e_8$	d_8^3	d_4^6	$\mathcal{G}$
$L_\Lambda(C)$	$D_5^2A_7^2$	D_{12}^2	D_6^4	A_3^8	D_8^3	D_4^6	A_1^{24}	Λ
$\tilde{V}_{L_\Lambda(C)}$	$D_{4,2}^2B_{2,1}^4$	$D_{6,1}^4$	$A_{3,1}^8$	$A_{1,2}^{16}$	$D_{4,1}^6$	$A_{1,1}^{24}$	V_Λ	$V^\natural$
$\dim V_1$	96	264	120	48	168	72	24	0