On the Feigin-Tipunin algebra

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September 22, 2021

Vertex operator algebra (VOA)

- VOA $=$ ∞-dim vector space $+$ complicated multiplication
- For a module M over a VOA, we have the conformal grading

$$M = \bigoplus_{\Delta \in \mathbb{Q}} M_\Delta$$

s.t. $L_0|_{M_\Delta} = \Delta$, $\dim M_\Delta < \infty$.

- VOA was first introduced in the 1980s for the study of monster groups and the mathematical formulation of 2d CFTs, and have since been applied to a wide range of fields including integrable systems, knot and 3-manifold theory, and modular forms.

Semisimple VOAs

- Traditionally, VOAs have been studied mainly in the semi-simple case, i.e., when all modules are decomposed into direct sum of irreducible modules.
- In this case the characters of irreducible modules give traditional modular forms.
- Also, its module category is a semi-simple modular tensor category, which gives a 3d TQFT.
- The theory of semi-simple VOAs is highly completed, and has many examples.

Log VOAs

- On the other hand, the study of non-semi-simple vertex algebras has also become popular in recent years.
- A log VOA is a non-semi-simple VOA which satisfies a certain finiteness condition called C_2-cofinite.
- The most famous, and almost the only known, examples of a log VOA a is the Feigin-Tipunin algebra of type A_1.
- The Feigin-Tipunin algebra of type A_1 has been studied by many mathematicians and its basic properties and representation theory are almost known.
- Moreover, it has been found that its characters coincide with Ramanujan's false theta function and also with certain knot invariants (Gukov's half index).

Problem and AIM

Problem

1. *The study of log VOAs is not explored as semi-simple cases, and only a few examples of log VOAs are known.*
2. *The FT algebra of type ADE has a very complicated structure, so algebraic and direct computational approaches are not applicable, and for a long time there has been no results on it.*

AIM

Develop theory on FT algebra of type ADE.

Overview

1. We prove the Feigin-Tipunin's conjecture on the geometric realization of the FT algebra of type ADE.
2. It enables us to use various geometric theorems.
3. By using the geometric realization, we prove the basic properties of the FT algebra of type ADE.
4. After that, for some special cases, we prove the C_2-cofiniteness of the FT algebra of type A_2.

Notations

- $\mathfrak{g}$: fin-dim simple Lie algebra of type ADE,
- G : corresponding algebraic group,
- $p \in \mathbb{Z}_{\geq 2}$,
- V_L : lattice VOA associated with $L = \sqrt{p}P$ or $\sqrt{p}Q$ (where Q : root lattice, P : weight lattice),
- $V_{L+\lambda}$: irreducible V_L-module (where $\lambda \in \Lambda = \frac{1}{\sqrt{p}}P/\sqrt{p}Q$)
- $\lambda \in \Lambda$ is decomposed as

$$\lambda = -\sqrt{p}\hat{\lambda} + \bar{\lambda},$$

  where
  - $\hat{\lambda}$: minuscule weight $\in P_+$,
  - $\bar{\lambda} \in \frac{1}{\sqrt{p}}P/\sqrt{p}P$.

$W \curvearrowright \Lambda$ and $F_{i,\lambda}$

Let W be the Weyl group of $\mathfrak{g}$. We have the action of W on Λ

$$\lambda \mapsto \sigma * \lambda = -\sqrt{p}\hat{\lambda} + \sigma(\bar{\lambda} + \frac{1}{\sqrt{p}}\rho) - \frac{1}{\sqrt{p}}\rho$$

for $\sigma \in W$, $\lambda \in \Lambda$.
For $1 \leq i \leq \operatorname{rank} \mathfrak{g}$, we have the operator

$$F_{i,\lambda} \colon V_{L+\lambda} \to V_{L+\sigma_i * \lambda},$$

where $\sigma_i \in W$ is the simple reflection.

Algebraic definition of FT algebra

Definition

The (algebraic definition of) FT algebra W_L^{alg} is given by

$$W_L^{\mathrm{alg}} = \bigcap_{\imath=1}^{\mathrm{rank}\,\mathfrak{g}} \ker F_{i,0}|_{V_L}.$$

We also have the W_L^{alg}-module $W_{L+\lambda}^{\mathrm{alg}}$ defined by

$$W_{L+\lambda}^{\mathrm{alg}} = \bigcap_{\imath=1}^{\mathrm{rank}\,\mathfrak{g}} \ker F_{i,\lambda}|_{V_{L+\lambda}}.$$

W_L^{alg} is a sub VOA of V_L and
$W_{L+\lambda}^{\mathrm{alg}}$ is a sub W_L^{alg}-module of $V_{L+\lambda}$.

Geometric definition of FT algebra

Let B be the Borel subgroup of G. We have

$$B \curvearrowright V_{L+\lambda}$$

and thus we have the homogeneous vector bundle $G \times_B V_{L+\lambda}$ over the flag variety G/B.

Definition

The (geometric definition of) FT algebra W_L is given by

$$W_L = H^0(G \times_B V_L).$$

We also have the W_L-module $W_{L+\lambda}$ defined by

$$W_{L+\lambda} = H^0(G \times_B V_{L+\lambda}).$$

Feigin-Tipunin's conjecture

Theorem (Feigin-Tipunin, S)

We have the VOA isomorphism

$$W_L \xrightarrow{\sim} W_L^{\mathrm{alg}},\ \ s \mapsto s(B)$$

More generally, we have the embedding

$$W_{L+\lambda} \hookrightarrow W_{L+\lambda}^{\mathrm{alg}},\ \ s \mapsto s(B),$$

where the $\hookrightarrow$ *above is* $\xrightarrow{\sim}$ *iff* λ *satisfies a certain easily checked condition.*
In particular, if $(\sqrt{p}\bar{\lambda} + \rho, \theta) \leq p$*, then* $W_{L+\lambda} \simeq W_{L+\lambda}^{\mathrm{alg}}$.

Proof

1. If $V \subseteq V_{L+\lambda}$ is a G-submodule of $V_{L+\lambda}$, then

$$V \simeq H^0(G \times_B V) \subseteq H^0(G \times_B V_{L+\lambda}) = W_{L+\lambda}.$$

   Thus $W_{L+\lambda}$ is the largest G-submodule of $V_{L+\lambda}$.
2. On the other hands, since

$$\ker F_{i,\lambda} \simeq H^0(P_i \times_B V_{L+\lambda}),$$

   $\ker F_{i,\lambda}$ is the largest P_i-submodule of $V_{L+\lambda}$.
3. Thus, we have

$$W_{L+\lambda} \subseteq W^{\mathrm{alg}}_{L+\lambda} = \bigcap_{i=1}^{\mathrm{rank}\,\mathfrak{g}} \ker F_{i,\lambda}|_{V_{L+\lambda}},$$

   and $W^{\mathrm{alg}}_{L+\lambda}$ has the G-module structure iff λ satisfies a certain easily checked condition.

$(\sqrt{p}\bar{\lambda} + \rho, \theta) \leq p$

From now on, let us assume that λ satisfies

$$(\sqrt{p}\bar{\lambda} + \rho, \theta) \leq p,$$

where ρ is the Weyl vector and θ is the highest root.
In particular, we assume that

$$p \geq h - 1.$$

Remark

The condition $(\sqrt{p}\bar{\lambda} + \rho, \theta) \leq p$ also appear in the modular representation theory of reduced enveloping algebras.

Duality theorem

Theorem (S)

For any $n \in \mathbb{Z}$, we have

$$H^n(G \times_B V_{L+\lambda}) \simeq H^{-n}(G \times_B V_{L-w_0(\lambda)})^*$$

as W_L- and G-modules,
where w_0 is the longest element of W and
$H^{-n}(G \times_B V_{L-w_0(\lambda)})^*$ means the restricted dual of
$H^{-n}(G \times_B V_{L-w_0(\lambda)})$ as W_L-module, i.e.,

$$H^{-n}(G \times_B V_{L-w_0(\lambda)})^* = \bigoplus_{\Delta \in \mathbb{Q}} H^{-n}(G \times_B (V_{L-w_0(\lambda)})_\Delta)^*.$$

Proof

1. By the Serre duality, we have

$$H^n(G \times_B V_{L+\lambda}) \simeq H^{\dim G/B-n}((G \times_B V_{L+\lambda})^* \otimes \mathcal{O}(-2\rho))^*,$$

   where $\mathcal{O}(-2\rho)$ is the line bundle over G/B.
2. On the other hands, by $(\sqrt{p}\bar{\lambda} + \rho, \theta) \leq p$, we obtain

$$H^{-n}(G \times_B V_{L-w_0(\lambda)}) \simeq H^{\dim G/B-n}((G \times_B V_{L+\lambda})^* \otimes \mathcal{O}(-2\rho))^*.$$

   in the same way as the proof of Borel-Weil-Bott's theorem.
3. By combining these two dualities, we obtain the assertion.

Duality for $n > 0$ (1)

Theorem (Feigin-Tipunin, S)

1. *We have the cohomology vanishing*

$$H^{n>0}(G \times_B V_{L+\lambda}) = 0.$$

2. *We heve the character formula*

$$\mathrm{ch}\, W_{L+\lambda} = \sum_{\alpha \in P_+ \cap \frac{1}{\sqrt{p}} L} \dim \mathcal{R}_{\alpha+\hat{\lambda}}\, \mathrm{ch}\, T^p_{\sqrt{p}\bar{\lambda}, \alpha+\hat{\lambda}},$$

by the Atiyah-Bott's fixed point theorem, where

- $\mathcal{R}_{\alpha+\hat{\lambda}}$ *: irreducible G-module with h.w $\alpha + \hat{\lambda}$.*
- $T^p_{\sqrt{p}\bar{\lambda}, \alpha+\hat{\lambda}}$ *: Arakawa-Frenkel module over the principal affine W-algebra $\mathcal{W}^{p-h}(\mathfrak{g})$ of level $p - h$.*

Duality for $n = 0$ (1)

Theorem (S)

We have the W_L- and G-module isomorphism

$$W_{L+\lambda} \simeq W^*_{L-w_0(\lambda)}.$$

In particular, W_L is simple and self-dual.

Since $W_{L+\lambda}$ has the G-action, we can decompose $W_{L+\lambda}$ as $G \times W_L^G$-module (where W_L^G is the G-orbifold of W_L).
By the character formula of W_L, we have

$$W_L^G \simeq \mathcal{W}^{p-h}(\mathfrak{g}).$$

Thus, we can decompose $W_{L+\lambda}$ as $G \times \mathcal{W}^{p-h}(\mathfrak{g})$-module.

Duality for $n = 0$ (2)

Theorem (S)

We have the $G \times \mathcal{W}^{p-h}(\mathfrak{g})$-module isomorphism

$$W_{L+\lambda} \simeq \bigoplus_{\alpha \in P_+ \cap \frac{1}{\sqrt{p}} L} \mathcal{R}_{\alpha+\hat{\lambda}} \otimes T^{p}_{\sqrt{p}\bar{\lambda}, \alpha+\hat{\lambda}}$$

Moreover, we have

$$\mathcal{W}^{p-h}(\mathfrak{g}) \simeq \mathcal{W}_{p-h}(\mathfrak{g}),$$

and $T^{p}_{\sqrt{p}\bar{\lambda}, \alpha+\hat{\lambda}}$ is the irreducible $\mathcal{W}_{p-h}(\mathfrak{g})$-module.

Remark

We can study the representation theory of non-generic W-algebras by using that of FT algebras!

Proof

1. Since W_L is simple, by the quantum Galois theory by [Dong-Li-Mason, McRae], we obtain the assertion for $\bar{\lambda} = 0$.
2. Since $T^p_{0,\alpha+\hat{\lambda}}$ is irreducible, we have

$$\begin{aligned} &\operatorname{ch} T^p_{0,\alpha+\hat{\lambda}} \text{ given in [Arakawa-Frenkel]} \\ =\ &\text{another character formula of the corresponding} \\ &\text{irreducible } \mathcal{W}_{p-h}(\mathfrak{g})\text{-module given in [Arakawa].} \end{aligned}$$

3. By $(\sqrt{p}\bar{\lambda} + \rho, \theta) \leq p$, this equation leads a certain relation which independent of the choice of λ.
4. Applying this relation generalize the simplicity of $T^p_{0,\alpha+\hat{\lambda}}$ to that of $T^p_{\sqrt{p}\bar{\lambda},\alpha+\hat{\lambda}}$.

Duality for $n = 0$ (3)

By combining the simplicity of $T^p_{\sqrt{p}\bar{\lambda},\alpha+\hat{\lambda}}$ and $W_{L+\lambda} \simeq W^*_{L-w_0(\lambda)}$, we obtain the following.

Theorem (S)

$W_{L+\lambda}$ is the irreducible W_L-module.

Remark

In the case of $\mathfrak{g} = \mathfrak{sl}_3$ and $p = 2$, before my work, the simplicity of W_L is proved in [Adamovic-Milas-Wang].

Conclusion

1. FT algebra is traditionally defined by the sub VOA $W_L^{\mathrm{alg}} = \bigcap_{i=1}^{\mathrm{rank}\,\mathfrak{g}} \ker F_{i,0}|_{V_L}$ of V_L.
2. Feigin-Tipunin conjectured W_L^{alg} has the geometric realization $W_L = H^0(G \times_B V_L)$, and proved in my first paper.
3. It enables us to use various strong geometric theorems, as Serre duality and Atiyah-Bott's theorem.
4. By using these theorems, when $(\sqrt{p}\bar{\lambda} + \rho, \theta) \leq p$, we proved the simplicity, character formula, and $G \times \mathcal{W}_{p-h}(\mathfrak{g})$-module structure of $W_{L+\lambda}$.
5. Also, as its corollary, we obtain the simplicity of non-generic Arakawa-Frenkel module $T^p_{\sqrt{p}\bar{\lambda}, \alpha+\hat{\lambda}}$.
   Moreover, $\mathrm{ch}\, W_{L+\lambda}$ is the higher rank version of the false theta function/Gukov's half index.

C_2-cofiniteness of W_L (1)

We want to show that W_L is C_2-cofinite, i.e.,

$$\dim R_{W_L} := \dim W_L/C_2(W_L) < \infty$$

Namely, we have to show that

1. R_{W_L} is finitely generated, and
2. All element in R_{W_L} is nilpotent.

The first problem is (probably) proved for $p \geq h - 1$.
(We use the simplicity of W_L and $T^p_{0,\alpha}$)
For the second problem, denote by

$$\pi \colon W_L \twoheadrightarrow R_{W_L}$$

the projection from W_L to R_{W_L}.

C_2-cofiniteness of W_L (2)

For the second problem, let us recall the decomposition

$$W_L \simeq \mathcal{W}_{p-h}(\mathfrak{g}) \oplus (\bigoplus_{\alpha \neq 0} \mathcal{R}_\alpha \otimes T^p_{0,\alpha})$$

1. It is easy to show that $\pi(\bigoplus_{\alpha \neq 0}) \subseteq R_{W_L}$ is nilpotent.
2. On the other hands, the principal affine W-algebra $\mathcal{W}_{p-h}(\mathfrak{g})$ itself is NOT C_2-cofinite, because

$$\mathcal{W}_{p-h}(\mathfrak{g}) \simeq \mathcal{W}^{p-h}(\mathfrak{g}).$$

3. Thus, to show that W_L is C_2-cofinite, we have to find enough additional relations from $\pi(\bigoplus_{\alpha \neq 0})$ to $\pi(\mathcal{W}_{p-h}(\mathfrak{g}))$.

However, direct calculation of the additional relations is almost impossible.

C_2-cofiniteness of W_L (3)

Theorem (S)

The additional relations from $\pi(\bigoplus_{\alpha\neq 0})$ *to* $\pi(\mathcal{W}_{p-h}(\mathfrak{g}))$ *are determined by a certain "algorithm" much easier than direct calculation.*

From now on, let us consider the case where $\mathfrak{g}=\mathfrak{sl}_3$. Denote by $\{L,W\}$ the generator of $\mathcal{W}^{p-h}(\mathfrak{sl}_3)$.

Example

When $p=2$*, by the algorithm, we have*

$$\pi((-\frac{1}{72}(L_{-2})^4+\frac{7}{16}L_{-2}(W_{-3})^2)|0\rangle) \text{ and}$$
$$\pi((-\frac{7}{12}(L_{-2})^3W_{-3}+\frac{21}{32}(W_{-3})^3)|0\rangle)$$

are nilpotent. By the Hilbert's Nullstellensatz, W_L *is* C_2*-cofinite.*

Example

When $p = 3$, by the algorithm, we have

$$\pi((-\frac{1}{35}(L_{-2})^7 + \frac{99}{32}(L_{-2})^4(W_{-3})^2 - \frac{81}{2}L_{-2}(W_{-3})^4)|0\rangle) \text{ and}$$
$$\pi((-\frac{687}{640}(L_{-2})^6 W_{-3} + \frac{621}{16}(L_{-2})^3(W_{-3})^3 - \frac{243}{4}(W_{-3})^5)|0\rangle)$$

are nilpotent. By the Hilbert's Nullstellensatz, W_L is C_2-cofinite.

Theorem (S)

When $\mathfrak{g} = \mathfrak{sl}_3$ and $p = 2, 3$, W_L is C_2-cofinite.

Future works (1)

1. When $W_{L+\lambda}$ is irreducible as W_L-module?
   - When $W_{L+\lambda} \simeq W^{\mathrm{alg}}_{L+\lambda}$?
   - For any $\lambda \in \Lambda$?
2. What is the multiplicities?
   - $[W^{\mathrm{alg}}_{L+\lambda} : W_{L+\sigma*\lambda}] =?$ $[V_{L+\lambda} : W_{L+\sigma*\lambda}] =?$
   - These multiplicities are given by a special value of some "good polynomial"?
3. log Kazhdan-Lusztig conjecture (when $\mathfrak{g} = \mathfrak{sl}_2$, proved in [Gannon-Negron])

   $$W_L - \mathrm{mod} \simeq \text{“}\mathfrak{u}_q(\mathfrak{g}) - \mathrm{mod}\text{”},$$

   where "$\mathfrak{u}_q(\mathfrak{g}) - \mathrm{mod}$" is a certain modification of the module category of the small quantum group at root of unity.
4. Relation between W_L and modular representation theory.

Future works (2)

1. general (non-principal) W-algebra version of the FT algebra. (In the case of affine VOA, studied by Adamovic et.al.)
2. Relationship between W_L-modules and knot theory/number theory (Gukov et.al).
   - $\sum a_\lambda \operatorname{ch} W_{L+\lambda} =$ "half index" of some Seifert 3-manifold?
   - Study the relation among

$$\text{3-manifold} \leftrightarrow \text{"spoiled" modular form} \leftrightarrow \text{VOA far from semisimple.}$$

   In particular,

   - For log VOAs, what kinds of 3-manifolds/"spoiled" modular forms corresponds?
   - For hyperbolic 3-manifolds, what kinds of VOAs/"spoiled" modular forms corresponds?

Thank you!

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