Rigidity in conformal field theory and (vertex) algebras beyond rigidity

[Allen, Fuchs, Lentner, Wood]

1. Rigidity

1.1 Def:

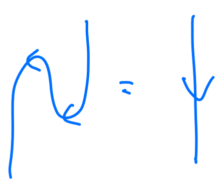
$(\mathcal{C}, \otimes, \mathbb{1})$ monoidal category, $c \in \mathcal{C}$. A right dual of c is an object

$c^\vee \in \mathcal{C}$ and

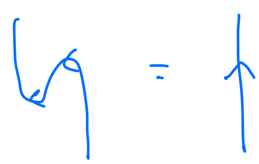
$ev^r : c^\vee \otimes c \to \mathbb{1}$ $\qquad$ $coev^r : \mathbb{1} \to c \otimes c^\vee$

s.t.



$id_{c^\vee}$



id_c

2. Left dual is defined analogously

3. $\mathcal{C}$ is rigid, if all objects have both left and right dual.

Rigidity is a property.

Examples

1. $\mathrm{vect}_{f.d.}$ $\quad ev^r : V^* \otimes V \to k$

$$\mathrm{coev}^r : k \longrightarrow \mathrm{End}(V) \cong V^* \otimes V$$
$$\lambda \longmapsto \lambda \, \mathrm{id}_V$$

2. Finitely generated proj. modules over a finite-dim. k-algebra are rigid.
3. H Hopf / field, $\dim H < \infty$. Then f.d. modules are rigid.
4. Category of tangles.

Consequences:

1. $\mathcal{C}$ finite tensor category $\Rightarrow$ tensor product exact
2. Dual is opmonoidal $\quad \mathcal{C} \longrightarrow \mathcal{C}^{opp, mopp}$

Vertex algebras : HLZ tensor product theory

Ex: $\quad W_{2,3} \quad c = 0 \qquad \otimes$ not exact.

What is a good notion of duality for vertex algebras?

2. A toy example

2.1 $\quad A$ k algebra, k a field, $\dim_k A < \infty$

$A\text{-bimod}_{f.d.}$ is monoidal, $\qquad B \otimes A \otimes \tilde{B} \to B \otimes \tilde{B} \to B \otimes_A \tilde{B}$

right exact $\qquad$ (coequalizer)

(in the sequel, every vector space will be finite-dimensional)

2.2 $G : A_1\text{-}A_2\text{-bimod} \longrightarrow A_2\text{-}A_1\text{-bimod}^{opp}$

$G(\beta)$ is $\beta^* = \mathrm{Hom}_k(B,k)$ as a vector space w/ actions
$(a_1 \beta a_2)(b) = \beta(a_2 b a_1)$

Observation: On $\mathcal{C}$ = A-bimod, G is not opmonoidal, in general

$$G(M \otimes_A N) \neq G(N) \otimes_A G(M)$$

Ex: $M = A$ $\quad G(N) \neq A^* \otimes_A G(N)$ $\quad$ Nakayama

Def

$$M \otimes^A N := G(GN \otimes_A GM)$$

is a second monoidal structure on $\mathcal{C}$

- monoidal unit is bimodule A^*
- $\otimes^A$ is an equalizer and hence left exact.

Exercise $\quad \mathrm{Hom}_{A\text{-bimod}}(M_1 \otimes M_2, A^*) \cong \mathrm{Hom}_{A\text{-bimod}}(M_1, G(M_2))$

A^* is a dualizing object for A-bimod (which is secretly a Hopf algebroid)

Rigidity not good for Hopf algebroids

2.3 Digression: the tensor product $\otimes^A$ is useful.

Eilenberg-Watts: $\quad F : \text{mod-}A_1 \to \text{mod-}A_2$ (finite-dimensional)
right exact, then

$$F(-) \cong - \otimes_{A_1} F(A_1)$$

Now let $H : A_1\text{-mod} \to A_2\text{-mod}$ left exact

$$\text{mod-}A_1 \xrightarrow{G} A_1\text{-mod} \xrightarrow{H} A_2\text{-mod} \xrightarrow{G} \text{mod-}A_1$$

is right exact and thus

$$G \circ H \circ G(-) \cong - \otimes_{A_1} GHG(A_1)$$

Thus applying to $\tilde{M} \in \text{mod-}A_1$ $\quad G(\tilde{M}) = M \in A_1\text{-mod}$, we get

$$HM \cong G^{-1}\left(GM \otimes_{A_1} GH(A_1^*) \right)$$

$$= H(A_1^*) \otimes^{A_1} M$$

(Eilenberg-Watts symmetric for left and right exact)

2.4 GV categories

Def: 1. $(\mathcal{C}, \otimes, a, l, r)$ monoidal. $K \in \mathcal{C}$ is called dualizing object, if the contravariant functor

$$X \longmapsto \mathrm{Hom}(X \otimes Y, K)$$

is representable by $G(Y) \in \mathcal{C}$ and contravariant functor

$$G : \mathcal{C} \to \mathcal{C}$$

is an anti-equivalence. Thus

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, K) \cong \mathrm{Hom}(X, G(Y))$$

2. A GV category is a monoidal category, together with the choice of a dualizing object.

Remarks

- GV categories are also known as $*$-autonomous categories (linearly distributive w/ negation)
- The choice of a dualizing object is structure (in contrast to duality).
- GV categories have many desirable properties. $\mathcal{C}$ abelian GV category
  - G^2 is monoidal
  - $\otimes$ is right exact, with internal Hom

$$\underline{\mathrm{Hom}}(X, Z) \cong G(X \otimes G^{-1} Z)$$

(CFT → boundary fields?)

  - $$X \cdot Y := G^{-1}(GY \otimes GX)$$

    is a left exact tensor product, admitting internal coHom's.
  - Notion of braided GV category exists
  - Notion of pivotal GV category exists
  - Notion of ribbon " " "

Thm [Müller-Woike, 2020]

- Cyclic algebras over the cyclic associative (1d) / framed little disc operad (2d) operad in FinCat are pivotal / balanced GV categories.

- Balanced braided GV lead to representations of handlebody groups, generalizing representations found in modular functors.

3. HLZ vertex algebras

3.1 V vertex algebra, $A \leq B$ abelian groups

- V graded by L_0 principal values and A ("root lattice")
- strongly B-graded "generalized" weak V-module M ("weight lattice")

$$M' = \bigoplus_{\substack{\beta \in B \\ h \in \mathbb{C}}} \left(M_h^{(\beta)} \right)^*$$

$*$ vector space dual ; is V-module contragredient module

$$\langle Y_{M'}(v,z)\phi, m \rangle = \langle \phi, Y_M^{opp}(v,z) m \rangle$$

with $Y_M^{opp}(v,z) := Y_M\left(e^{zL_1}(-z^{-2})^{L_0} v, z^{-1} \right)$

depends on conformal structure

- Fact $M'' \cong M$
- V' and V not necess. isomorphic (ex: $W_{2,3}$)

$\longrightarrow$ Categorical role for V' ?

- Contragredients are important for HLZ tensor product theory

$$M_1 \boxtimes M_2 \subset \mathrm{COMP}(M_1, M_2) \subset (M_1 \otimes M_2)^*$$

↑ good module

$$M_1 \otimes M_2 = (M_1 \boxtimes M_2)'$$

The tensor product which is a subspace comes first.

- HLZ prove iso of dual intertwiners for strongly $\mathbb{Z}$-graded generalized modules

$$I\binom{M_3}{M_1\ M_2} \cong I\binom{M_2'}{M_1\ M_3'}$$

3.2 Condense these insights to

<u>Thm</u> [ALSW 2021]

$\mathcal{C} = V\text{-mod}$ ribbon GV category. V' is a dualizing object.

„Pf"

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X \otimes Z', Y')$$

$$\mathrm{Hom}(X \otimes Y, V') \cong \mathrm{Hom}(X \otimes V, Y') = \mathrm{Hom}(X, Y')$$

□

3.3 The Feigin-Fuchs boson provides examples of GV categories.

Different Conformal vectors (structure!) for Heisenberg algebras for $\mathfrak{h}$

$$\omega_\gamma = \frac{1}{2} \sum_i \alpha^i_{-1} \alpha^i_{-1} |0\rangle + \gamma_{-2} |0\rangle$$

for suitable $\gamma \in \mathfrak{h}$

Simplest case : even lattice L
- $A = L^*/L$ finite abelian group, take $A = \mathbb{Z}_N$
- $(\Omega, F) \in H^3_{ab}(A, \mathbb{C}^*)$

$\rightsquigarrow$ ssi braided tensor cat $\mathcal{C}_{(A,F,\Omega)}$ with

fusion ring $\mathbb{Z}A$; all objects invertible.

<u>Thm</u>

- All simple objects can be chosen as dualizing objects
  - N even, $k \in \mathbb{Z}_N$ even $\rightsquigarrow$ ribbon GV.

4. Full CFT

4.1 <u>Q</u> : Given $\mathcal{C} = V\text{-rep}$ a ribbon GV category, can we construct a full, local CFT?
(includes boundary fields, bulk field obeying modular invariance, OPE's)

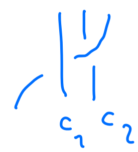
Problems

1) Non-degeneracy of the braiding

Particularly useful characterization: braided monoidal category is factorizable, if canonical braided monoidal functor

$$\Xi : \mathcal{C} \boxtimes \mathcal{C}^{rev} \longrightarrow Z(\mathcal{C})$$

$$c_1 \boxtimes c_2 \longmapsto c_1 \otimes c_2$$



is an equivalence.

- Allows to construct bulk fields as objects in $Z(\mathcal{C})$ [Fuchs, CS]
- Allows to use stringnet techniques ("skein theoretic") [Fuchs, CS, Yang]

Problem: Even if $\mathcal{C}$ is finitely ssi, $Z(\mathcal{C})$ can be infinite and non semi simple (→ no known skein techniques)

2) Theory of module categories („different modular invariants") breaks down

Problem: Even if $\mathcal{C}$ is finitely ssi, it can have infinitely many(*) module categories. Relative Deligne product unclear. Morita 3-category of GV categories.

(*) inequivalent indecomposable

4.2. The following successes in the rigid case can therefore not be implemented directly in the GV case.

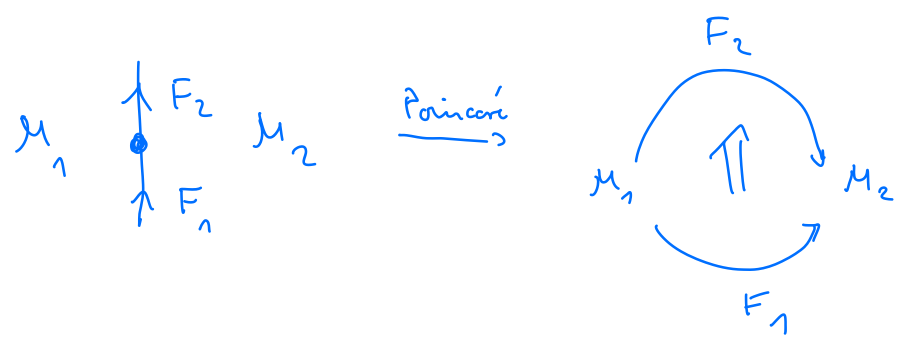
Focus on field content

$\mathcal{C}$ = (rigid) modular tensor category

${}_{\mathcal{C}}\mathcal{M}$ exact $\mathcal{C}$-module = specifies full CFT

$F : {}_{\mathcal{C}}\mathcal{M} \to {}_{\mathcal{C}}\mathcal{N}$ module functor = top. defect line

Fields



- $\mathrm{Rex}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ exact $Z(\mathcal{C})$-module

Inner Hom $\underline{\mathrm{Nat}}(F_1, F_2) \in Z(\mathcal{C})$ describes field content.

- Thm [Fuchs, CS, 2021]
  $\mathcal{M}_1, \mathcal{M}_2$ pivotal
  $\mathcal{C}$ unimodular
  $\Rightarrow$ $\mathrm{Rex}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ pivotal

In particular:

- Nat (F, F) is symmetric Frobenius algebra
- "Eckmann-Hilton"

  Nat (id, id) commutative sym. FA (bulk fields)

Thm

$$\mathrm{Nat}(F,G) = \int_{m_1 \in \mathcal{M}_1} \mathrm{Hom}(F(m_1), G(m_1))$$

⇝ Cardy case: $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{C}$, bulk fields

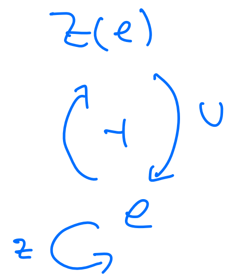
$$\mathrm{Nat}(id, id) = \int_{c \in \mathcal{C}} \mathrm{Hom}(c, c) \in Z(\mathcal{C})$$

$$= \int^{c \in \mathcal{C}} c \otimes {}^{\vee}c$$

- Horizontal and vertical composition give OPE's satisfying genus zero constraints of Lewellen.

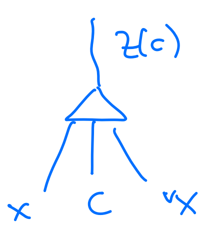
4.3 For GV, we only have repns of handlebody groups.

Indeed, the modular functor of Lyubashenko is based on the central monad

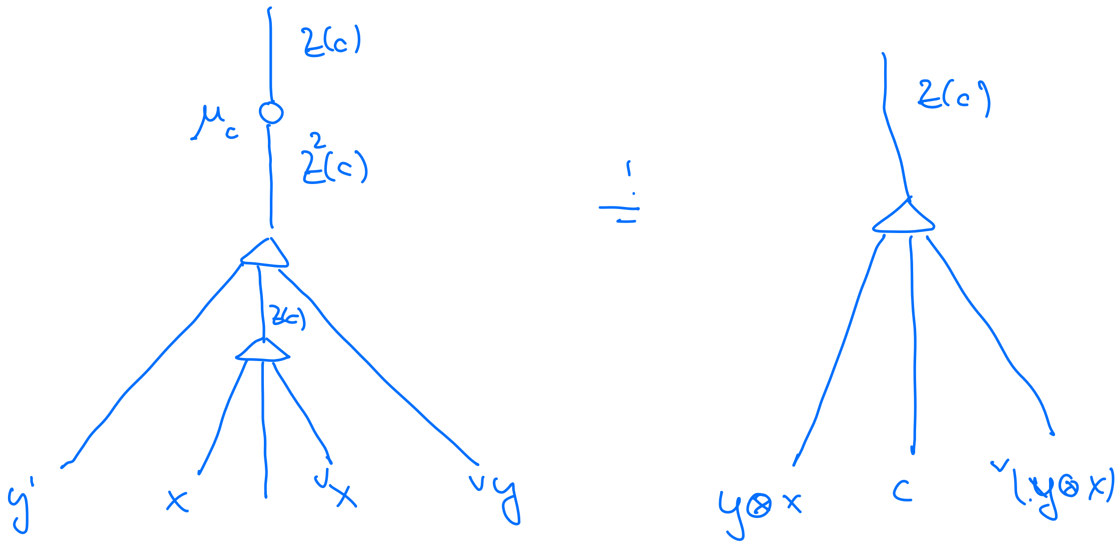


$$Z(c) = \int^{x\in\mathcal{C}} \underline{\mathrm{Hom}}(x, x\otimes c)$$

But multiplication of the monad: $\mu: Z^2 \Rightarrow Z$ is constructed for $\mathcal{C}$ rigid using that dual is op monoidal:



structure morphism



5. Conclusions

- We need to understand better categorical properties of GV categories from HLZ vertex algebras

- We need to go beyond finite tensor categories and exact module categories.