TECHNISCHE UNIVERSITÄT DARMSTADT

$\beta\gamma$ ghosts: Whittaker modules and fusion rules

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QuantiXLie
PROVEDBA VRHUNSKIH ISTRAŽIVANJA U SKLOPU
ZNANSTVENOG CENTRA IZVRSNOSTI
ZA KVANTNE I KOMPLEKSNE SUSTAVE
TE REPREZENTACIJE LIEJEVIH ALGEBRI

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This talk is based on two papers:

- [AP19] D. Adamović, V. Pedić, On fusion rules and intertwining operators for the Weyl vertex algebra, Journal of Mathematical Physics, 60(8), 2019.
- [ALPY19] D. Adamović, C. H. Lam, V. Pedić, N. Yu, On irreducibility of modules of Whittaker type for cyclic orbifold vertex algebras, Journal of Algebra, 539, 2019

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Two important problems in VOA theory

- Fusion rules
  - Defined in [FHL93] as the dimensions of vector spaces of intertwining operators.
  - Y. Z. Huang proved that Verlinde formula works for rational VOAs.
  - Still open problem for most of the irrational examples.
  - Some new results:
    - Singlet vertex algebra $\mathcal{M}(p)$ associated to $(1, p)$–modules of Virasoro algebra: Adamović, Milas, Creutzig, McRae, Yang
    - Affine vertex algebra: Adamović, Milas, Creutzig, Ridout
    - Weyl vertex algebra: Ridout, Wood, Allen
  - In this talk we present the proof of the Verlinde formula for the case of Weyl vertex algebra.

Two important problems in VOA theory

- Orbifold problem
  - One of the main tools in the construction of new vertex algebras
  - We take a vertex algebra V and its finite automorphism group G, and we study the representation theory of the orbifold subalgebra V^G
  - Initiated in the 90sin papers [DM97], [DLM00] by C. Dong and G. Mason. However, the quantum Galois theory they founded relies on the $\mathbb{Z}$–gradation of modules and uses Zhu theory. Therefore, it can not be applied to weak modules, such as Whittaker modules.
  - Some new results
    - affine vertex algebra - Adamović, Lu, Zhao
    - Virasoro vertex algebra - Mazorchuk, Zhao, Lu, Ondrus, Wiesner
    - Heisenberg vertex algebra - Yu, Hartwig, Tanabe
  - In this talk we present an extension of Dong-Mason results to the category of weak modules
  - Also, we construct a counterexample to generalisation of the previous result to the case of infinite automorphism group.

Vertex algebra

Vertex algebra is a vector space V together with a distinguished vector $\mathbf{1}$ called the *vacuum vector* and endowed with a linear map Y such that

$$Y : V \to (\mathrm{End}\ V)[[z, z^{-1}]],$$
$$v \mapsto \sum_{n\in\mathbb{Z}} v_n z^{-n-1} = Y(v, z),$$

and for all $a, b \in V$ the following axioms hold:

- (truncation) $a_n b = 0$, for $n \gg 0$,
- (vacuum) $Y(\mathbf{1}, z) = \mathrm{Id}$, i. e. $\mathbf{1}_n = \delta_{n,-1}\mathrm{Id}$,
- (creation) $Y(a, z)\mathbf{1} \in V[[z]]$ & $\lim_{z\to 0} Y(a, z)\mathbf{1} = a$,
- (Jacobi)

$$z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y(u, z_1)Y(v, z_2) - x_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)Y(v, z_2)Y(u, z_1)$$
$$= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)Y(Y(u, z_0)v, z_2)$$

Vertex operator algebra

Vertex operator algebra (VOA) is a vertex algebra such that the underlying vector space is $\mathbb{Z}$-graduated by weights of vectors:

$$V = \bigoplus_{n \in \mathbb{Z}} V_{(n)},$$

- $\dim V_{(n)} < \infty$, for $n \in \mathbb{Z}$ and $V_{(n)} = 0$, for a small enough n,
- $\mathbf{1} \in V_{(0)}$,

together with a distinguished vector $\omega \in V_{(2)}$, called the *Virasoro vector*.

Also, if we define $L(n) = \omega_{n+1}$, for $n \in \mathbb{Z}$,

- Virasoro algebra relations hold:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}c\mathbf{1},$$

- $L(0)v = nv = (\text{wt } v)v$, for $v \in V_{(n)}$,
- $\frac{d}{dx}Y(v,x) = Y(L(-1)v, x)$.

Weak and ordinary VOA modules

Weak V–module is a complex vector space W, together with a linear map

$$Y_W : \ V \to \mathrm{End}(W)[[z, z^{-1}]],$$
$$a \mapsto Y_W(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

such that all the VA axioms which make sense hold.

A weak module (W, Y_W) is called an ordinary module if:

- (the $L(-1)$ derivative property): for any $a \in V$

$$Y_W(L(-1)a, z) = \frac{d}{dz} Y_W(a, z)$$

- (the grading property): $W = \bigoplus_{\alpha \in \mathbb{C}} W(\alpha)$, where

$$W(\alpha) = \{v \in W \mid L(0)v = \alpha v\},$$

  such that for every α, $\dim\,(\alpha) < \infty$ and $W(\alpha + n) = 0$ for sufficiently negative n.

Affine vertex (super)algebra

- $\mathfrak{g}$ complex simple Lie superalgebra with nondegenerate supersymmetric invariant bilinear form $(\cdot,\cdot)$
- $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ affine Kac-Moody superalgebra
- $\mathfrak{p} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c \leq \widehat{\mathfrak{g}}$, for every $k \in \mathbb{C}$, every $\mathfrak{g}$–module U becomes a $\mathfrak{p}$–module with: $cv = kv$, $(\mathfrak{g} \otimes t\mathbb{C}[t])v = 0$, $v \in U$.
- Induced $\widehat{\mathfrak{g}}$–module (generalised Verma module):

$$N_{\widehat{\mathfrak{g}}}(k, U) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{p})} U,$$

  where $U(\widehat{\mathfrak{g}})$ is the universal enveloping algebra of $\widehat{\mathfrak{g}}$
- For $U = \mathbb{C}v_0$, universal affine vertex algebra associated with $\mathfrak{g}$ on level k:

$$V^k(\mathfrak{g}) = N_{\widehat{\mathfrak{g}}}(k, 0)$$

- If k is non-critical, the unique irreducible quotient $L_k(\mathfrak{g})$.
- Example: $L_1(\mathfrak{g})$, associated to the Lie superalgebra $\mathfrak{gl}(1|1)$.

$\beta\gamma$ ghost (= Weyl vertex algebra)

Weyl algebra $\widehat{\mathcal{A}}$ is an associative algebra with generators $a(n), a^*(n)$, $n \in \mathbb{Z}$, and relations (for $n, m \in \mathbb{Z}$)

$$[a(n), a^*(m)] = \delta_{n+m,0}, \quad [a(n), a(m)] = [a^*(m), a^*(n)] = 0.$$

Let M be an irreducible $\mathcal{A}$–module generated by a cyclic vector $\mathbf{1}$ such that

$$a(n)\mathbf{1} = a^*(n+1)\mathbf{1} = 0. \quad (n \geq 0).$$

As a vector space,

$$M \;\cong\; \mathbb{C}[a(-n), a^*(-m) \mid n > 0,\; m \geq 0].$$

Then there is a unique vertex algebra $(M, Y, \mathbf{1})$, such that the vertex operator is defined as $Y : M \to \mathrm{End}(M)[[z, z^{-1}]]$,

$$Y(a(-1)\mathbf{1}, z) = a(z), \qquad Y(a^*(0)\mathbf{1}, z) = a^*(z),$$

$$a(z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}, \qquad a^*(z) = \sum_{n\in\mathbb{Z}} a^*(n) z^{-n}.$$

$\beta\gamma$ ghost: notes on representation theory

Let $\beta = a(-1)a^*(0)\mathbf{1}$ be a Heisenberg vector of level 1 with the corresponding field $\beta(z) = \sum_{n\in\mathbb{Z}} \beta(n)z^{-n-2}$.
Vertex algebra M also admits a family of Virasoro vectors

$$\omega_\mu = (1-\mu)a(-1)a^*(-1)\mathbf{1} - \mu a(-2)a^*(0)\mathbf{1} \quad (\mu \in \mathbb{C})$$

of central charge $c_\mu = 2(6\mu(\mu-1)+1)$.
We fix $\mu = 0$ and denote $L(n) = L^0(n)$, $c = c_0 = 2$.
We say that a module W for Weyl vertex algebra M is a weight module if the operators $\beta(0)$ and $L(0)$ act semi-simply on W.

$\beta\gamma$ ghost: notes on representation theory

Whittaker module for the Weyl algebra $\widehat{\mathcal{A}}$ is the quotient

$$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \widehat{\mathcal{A}}/I,$$

where $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_n)$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ and I is the left ideal

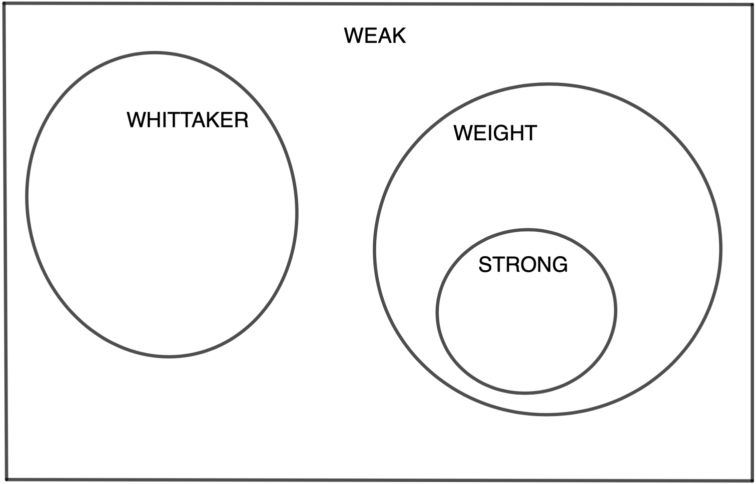
$$I = \langle a(0) - \lambda_0, \dots, a(n) - \lambda_n, a^*(1) - \mu_1, ..., a^*(n) - \mu_n, a(n+1), a^*(n+1), .. \rangle.$$

Proposition

(i) $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible $\widehat{\mathcal{A}}$–module.
(ii) $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible module for the Weyl vertex algebra M.

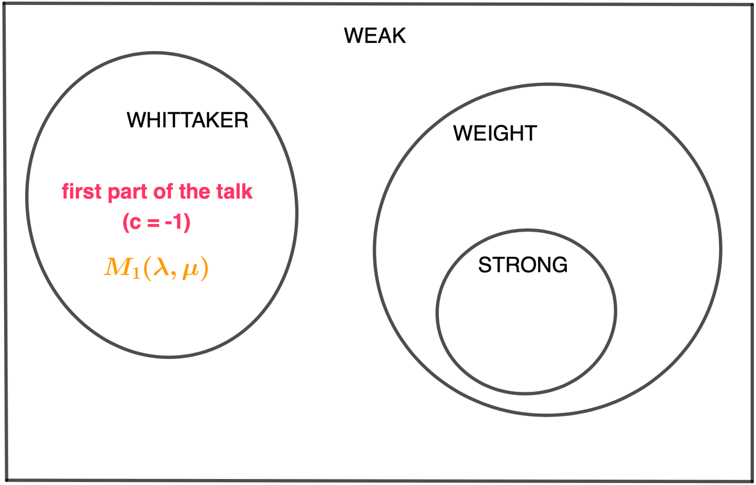
$\beta\gamma$ ghost: notes on representation theory

βγ ghost modules

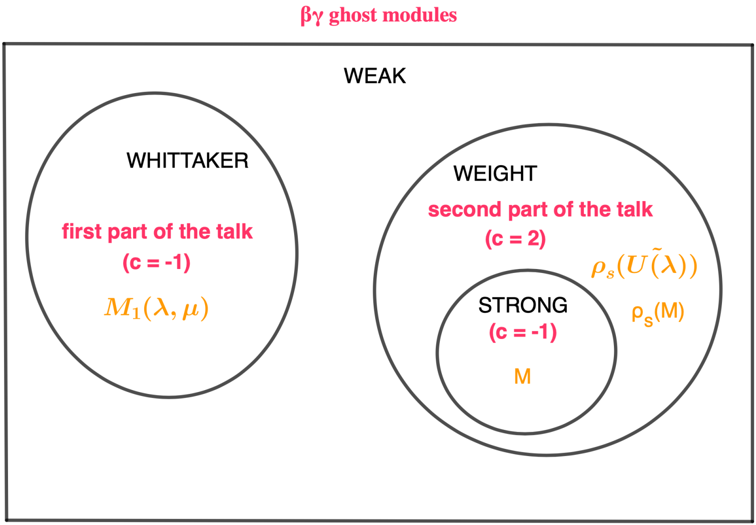


$\beta\gamma$ ghost: notes on representation theory

βγ ghost modules



$\beta\gamma$ ghost: notes on representation theory



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Intertwining operators

- Let M_1, M_2, M_3 be V–modules. An intertwining operator of type $\binom{M_3}{M_1\ M_2}$ is a map $I : a \mapsto I(a,z) = \sum_{n\in\mathbb{Z}} a^I_{(n)} z^{-n-1}$ from the module M_1 to the space of fields with coefficients in $Hom(M_2, M_3)$ such that:
  1. for $a \in V$, $b \in M_1$, $c \in M_2$, the following Jacobi identity holds:
  $$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{M_3}(a,z_1) I(b,z_2)c - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) I(b,z_2) Y_{M_2}(a,z_1)c$$
  $$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) I(Y_{M_1}(a,z_0)b, z_2)c,$$
  2. for all $a \in M_1$
  $$I(L(-1)a, z) = \frac{d}{dz} I(a,z).$$
- $I\binom{M_3}{M_1\ M_2}$ = space of interwining operators of type $\binom{M_3}{M_1\ M_2}$
- $N^{M_3}_{M_1,M_2} = \dim I \begin{pmatrix} M_3 \\ M_1\ \ M_2 \end{pmatrix}$ = **fusion coefficient** (when finite).

Intertwining operators

Proposition

Let g be a VOA V automorphism such that the following conditions hold:

$$\omega - g(\omega) \in Im(D), \quad \omega - g^{-1}(\omega) \in ImD. \tag{1}$$

Let M_1, M_2, M_3 be V–modules and let $I(\cdot, z)$ be an intertwining operator of type $\binom{M_3}{M_1 \ M_2}$. Then there is an intertwining operator I^g of type $\binom{M_3^g}{M_1^g \ M_2^g}$, such that $I^g(b, z_1) = I(b, z_1)$, for all $b \in M_1$. Also,

$$N_{M_1, M_2}^{M_3} = N_{M_1^g, M_2^g}^{M_3^g}.$$

- If V is a VOA and g an automorphism of V, then $g(\omega) = \omega$ (i $g^{-1}(\omega) = \omega$) by definition, so the proposition immediately holds.
- In our examples, g is only a vertex algebra automorphism, so in general $g(\omega) \neq \omega$. However, we show that the proposition holds.

Fusion rules

Let $\mathcal{K}$ be a category of $L(0)$–semisimple V–modules. Let M_1, M_2 and W_i, $i = 1, \dots, n$ be irreducible modules in category $\mathcal{K}$. We say that the fusion rule

$$M_1 \times M_2 = \sum_{i=1}^{n} W_i$$

holds in $\mathcal{K}$ if $N_{M_1, M_2}^{W_i} = 1$, $i = 1, \dots, n$, and $N_{M_1, M_2}^{R} = 0$ for every other irreducible V–module R in $\mathcal{K}$, which is not isomorphic to any of the modules W_i, $i = 1, \dots, n$.

$\beta\gamma$ ghost: module category

- We now wish to describe a category of modules for which we prove the fusion rules results
- For every $s \in \mathbb{Z}$, we can define the following automorphism for the Weyl algebra $\widehat{\mathcal{A}}$:

$$\rho_s(a(n)) = a(n+s), \quad \rho_s(a^*(n)) = a^*(n-s).$$

  We call it the spectral flow automorphism.
- The first Weyl algebra A_1 is generated by x, ∂_x together with commutation relations $[\partial_x, x] = 1$.
- For every $\lambda \in \mathbb{C}$,

$$U(\lambda) = x^\lambda \mathbb{C}[x, x^{-1}]$$

  is an A_1–module.

$\beta\gamma$ ghost: module category

- We define the following subalgebras of $\widehat{\mathcal{A}}$:

$$\widehat{\mathcal{A}}_{\geq 0} = \mathbb{C}[a(n), a^*(m) \mid n, m \in \mathbb{Z}_{\geq 0}],$$

$$\widehat{\mathcal{A}}_{<0} = \mathbb{C}[a(-n), a^*(-n) \mid n \in \mathbb{Z}_{\geq 1}].$$

Then $U(\lambda)$ is an $\widehat{\mathcal{A}}_{\geq 0}$–module if we define $a(n) = a^*(n) = 0$, on $U(\lambda)$, for every $(n \geq 1)$. Now we have the induced left $\widehat{\mathcal{A}}$–module:

$$\widetilde{U(\lambda)} = \widehat{\mathcal{A}} \otimes_{\widehat{\mathcal{A}}_{\geq 0}} U(\lambda)$$

isomorphic to:

$$\mathbb{C}[a(-n), a^*(-n) \mid n \geq 1] \otimes U(\lambda)$$

as a vector space.

$\beta\gamma$ ghost: module category

Proposition (A)

For every $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, $\widetilde{U(\lambda)}$ is an irreducible weight module for M.

Corollary (B)

Let g be an automorphism of Weyl vertex algebra M:

$$a \mapsto -a^*, \quad a^* \mapsto a.$$

For every $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ and $s \in \mathbb{Z}$:

$$\widetilde{U(\lambda)}^{g} \cong \rho_1(\widetilde{U(-\lambda)}), \quad \left(\rho_{-s+1}(\widetilde{U(\lambda)})\right)^{g} \cong \rho_s(\widetilde{U(-\lambda)}).$$

Main theorem

Let $\mathcal{K}$ be the category of weight M–modules such that the operators $\beta(n)$, $n \geq 1$, act locally nilpotent on every module in the category $\mathcal{K}$.

Teorem (AP)

Let $\lambda, \mu, \lambda + \mu \in \mathbb{C} \setminus \mathbb{Z}$. *Then:*

(i) $\rho_{\ell_1}(M) \times \rho_{\ell_2}(M) = \rho_{\ell_1+\ell_2}(M)$,

(ii) $\rho_{\ell_1}(M) \times \rho_{\ell_2}(\widetilde{U(\lambda)}) = \rho_{\ell_1+\ell_2}(\widetilde{U(\lambda)})$,

(iii) $\rho_{\ell_1}(\widetilde{U(\lambda)}) \times \rho_{\ell_2}(\widetilde{U(\mu)}) = \rho_{\ell_1+\ell_2}(\widetilde{U(\lambda+\mu)}) + \rho_{\ell_1+\ell_2-1}(\widetilde{U(\lambda+\mu)})$.

Main theorem

Verlinde type conjecture for $\beta\gamma$ ghost fusion rules was given by D. Ridout and S. Wood in Lett. Math. Phys. (2015). Our result is a proof of this conjecture.
Statements (i) and (ii) of the main theorem are easily proved by using H. Li results. To show the part (iii) of our theorem, it is equivalent to show:

(i) $\dim I \begin{pmatrix} \rho_s(\widetilde{U(\lambda)}) \\ \rho_{s_1}(\widetilde{U(\lambda_1)}) \ \rho_{s_2}(\widetilde{U(\lambda_2)}) \end{pmatrix} \leq 1,$

(ii) $\dim I \begin{pmatrix} \rho_s(\widetilde{U(\lambda)}) \\ \rho_{s_1}(\widetilde{U(\lambda_1)}) \ \rho_{s_2}(\widetilde{U(\lambda_2)}) \end{pmatrix} = 1 \iff \lambda = \lambda_1 + \lambda_2,$ $s = s_1 + s_2,$

or $\lambda = \lambda_1 + \lambda_2$, $s = s_1 + s_2 - 1$.

Construction of intertwining operators

Here we construct the intertwining operators using lattice vertex algebra.

- Let L be the lattice

$$L = \mathbb{Z}\alpha + \mathbb{Z}\beta, \ \langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1, \quad \langle \alpha, \beta \rangle = 0,$$

  and let $V_L = M_{\alpha,\beta}(1) \otimes \mathbb{C}[L]$ be the vertex superalgebra associated to this lattice,where $M_{\alpha,\beta}(1)$ is the Heisenberg vertex algebra generated by fields $\alpha(z)$ i $\beta(z)$, and $\mathbb{C}[L]$ is the group algebra. Let us define its vertex subalgebra:

$$\Pi(0) = M_{\alpha,\beta}(1) \otimes \mathbb{C}[\mathbb{Z}(\alpha + \beta)] \subset V_L.$$

- Then we have an injective homomorphism of vertex algebras $f : M \to \Pi(0)$ such that

$$f(a) = e^{\alpha+\beta}, \ f(a^*) = -\alpha(-1)e^{-\alpha-\beta}.$$

  Identifying a and a^* with their images in $\Pi(0)$, we get the embedding of M into $\Pi(0)$.

Construction of intertwining operators

- We define irreducible $\Pi(0)$–modules:

$$\Pi_r(\lambda) = \Pi(0).e^{r\beta+\lambda(\alpha+\beta)}.$$

Using lattice VOA results (Dong-Lepowsky), we obtain an intertwining operator of type:

$$\binom{\Pi_{r_1+r_2}(\lambda+\mu)}{\Pi_{r_1}(\lambda)\ \Pi_{r_2}(\mu)}$$

for vertex algebra $\Pi(0)$, and we take its restriction to M.

Proposition

Let $\ell \in \mathbb{Z}$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Then the following M–module isomorphisms hold:

(1) $\Pi_\ell(\lambda) \cong \rho_{-\ell+1}(\widetilde{U(-\lambda)})$,

(2) $\Pi_\ell(\lambda)^g \cong \rho_\ell(\widetilde{U(\lambda)})$.

Construction of intertwining operators

- Using previous proposition, we calculate:

$$I\begin{pmatrix} \Pi_{r_1+r_2}(\lambda+\mu) \\ \Pi_{r_1}(\lambda)\ \Pi_{r_2}(\mu) \end{pmatrix} = I\begin{pmatrix} \rho_{(-r_1-r_2+1}(U(\widetilde{-\lambda-\mu)}) \\ \rho_{-r_1+1)}(\widetilde{U(-\lambda)})\ \rho_{-r_2+1}(\widetilde{U(-\mu)}) \end{pmatrix}$$

$$= I\begin{pmatrix} \rho_{l_1+l_2-1}(U(\widetilde{-\lambda-\mu)}) \\ \rho_{l_1}(\widetilde{U(-\lambda)})\ \rho_{l_2}(\widetilde{U(-\mu)}) \end{pmatrix} \cong I\begin{pmatrix} \rho_{l_1+l_2-1}(U(\widetilde{\lambda+\mu)}) \\ \rho_{l_1}(\widetilde{U(\lambda)})\ \rho_{l_2}(\widetilde{U(\mu)}) \end{pmatrix},$$

  and so we have constructed an intertwining operator in the category of M–modules.

- Taking automorphism g form corrolary B and using proposition A, we get a new intertwining operator:

$$\begin{pmatrix} \rho_{l_1+l_2}(U(\widetilde{\lambda+\mu)}) \\ \rho_{l_1}(\widetilde{U(\lambda)})\ \rho_{l_2}(\widetilde{U(\mu)}) \end{pmatrix}.$$

Vertex algebra $L_1(\mathfrak{gl}(1|1))$

- To calculate fusion rules, we use affine Lie superalgebra $\mathfrak{gl}(1|1)$ and its associated affine vertex algebra $L_1(\mathfrak{gl}(1|1))$, because fusion rules are known for the latter (Creutzig-Ridout, J. Phys. A 46 (2013)).
- Let $\mathfrak{g} = \mathfrak{gl}(1|1)$ be the complex Lie superalgebra generated by two even elements E and N,two odd elements Ψ^+, Ψ^-, and by the following (super)commutation relations:

$$[\Psi^+, \Psi^-] = E, \ [E, \Psi^\pm] = [E, N] = 0, \ [N, \Psi^\pm] = \pm\Psi^\pm.$$

- Let $\hat{\mathfrak{g}} = \widehat{gl(1|1)} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the associated affine Lie superalgebra with commutation relations:

$$[x(n), y(m)] = [x, y](n + m) + n\delta_{n+m,0}(x|y)K,$$

where K is central, and we write $x(n) = x \otimes t^n$, for $x \in \mathfrak{g}$.
Let $L_k(\mathfrak{g})$ be its associated simple affine vertex algebra of rank k.

Vertex algebra $L_1(\mathfrak{gl}(1|1))$

- Let $\mathcal{V}_{r,s}$ be the Verma module for the Lie superaglebra $\mathfrak{g}$, generated by vector $v_{r,s}$, such that $Nv_{r,s} = rv_{r,s}$, $Ev_{r,s} = sv_{r,s}$.
- We use the following tensor product decomposition:

$$\mathcal{V}_{r_1,s_1} \otimes \mathcal{V}_{r_2,s_2} = \mathcal{V}_{r_1+r_2,s_1+s_2} \oplus \mathcal{V}_{r_1+r_2-1,s_1+s_2} \quad (s_1+s_2 \neq 0),$$

- Let $\widehat{\mathcal{V}}_{r,s}$ denote the rank 1 Verma module induced by the irreducible $\mathfrak{gl}(1|1)$–module $\mathcal{V}_{r,s}$.

Proposition

Let $r_1, r_2, s_1, s_2 \in \mathbb{C}$, $s_1, s_2, s_1+s_2 \notin \mathbb{Z}$. Then: $\dim I\binom{\widehat{\mathcal{V}}_{r_3,s_3}}{\widehat{\mathcal{V}}_{r_1,s_1} \ \widehat{\mathcal{V}}_{r_2,s_2}} \leq 1$.

Assume there is a non-trivial IO $\binom{\widehat{\mathcal{V}}_{r_3,s_3}}{\widehat{\mathcal{V}}_{r_1,s_1} \ \widehat{\mathcal{V}}_{r_2,s_2}}$ in the category of $L_1(\mathfrak{g})$–modules. Then $s_3 = s_1+s_2$ and $r_3 = r_1+r_2$, or $r_3 = r_1+r_2-1$.

Vertex algebra $L_1(\mathfrak{gl}(1|1))$

- Clifford vertex algebra F is generated by fields:

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi(n+\frac{1}{2})z^{-n-1},$$
$$\psi^*(z) = \sum_{n\in\mathbb{Z}} \psi^*(n+\frac{1}{2})z^{-n-1}.$$

  As a vector space,

$$F \cong \bigwedge \left(\{\psi(r), \psi^*(s) \mid r, s < 0\}\right)$$

- Let us define the following vertex superalgebra:

$$S\Pi(0) = \Pi(0) \otimes F \subset V_L,$$

  and its irreducible modules:

$$S\Pi_r(\lambda) = \Pi_r(\lambda) \otimes F = S\Pi(0).e^{r\beta+\lambda(\alpha+\beta)}.$$

Vertex algebra $L_1(\mathfrak{gl}(1|1))$

- Let $\mathcal{U} = M \otimes F$ and let us define vectors:

$$\Psi^+ := e^{\alpha+\beta+\gamma} = a(-1)\psi, \ \Psi^- := -\alpha(-1)e^{-\alpha-\beta-\gamma} = a^*(0)\psi^*,$$
$$E := \gamma + \beta, \ N := \frac{1}{2}(\gamma - \beta).$$

  Then $\mathcal{U}$ is a $\hat{\mathfrak{g}}$–module of level 1.

- We have the following gradation:

$$\mathcal{U} = \bigoplus \mathcal{U}^\ell, \quad E(0)|_{\mathcal{U}^\ell} = \ell \ \mathrm{Id}.$$

Proposition (Kac)

We have:

$$L_1(\mathfrak{g}) \cong \mathcal{U}^0 = Ker_{M\otimes F} E(0).$$

Calculation of fusion rules

Let us finish the fusion rules calculations for M.

Teorem

Let $r \in \mathbb{Z}$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Then:

(i) $S\Pi_r(\lambda)$ *is an irreducible* $M \otimes F$*–module,*

(ii) $S\Pi_r(\lambda)$ *is a completely reducible* $\widehat{gl(1|1)}$*–module:*

$$S\Pi_r(\lambda) \cong \bigoplus_{s\in\mathbb{Z}} U(\hat{\mathfrak{g}}).e^{r(\beta+\gamma)+(\lambda+s)(\alpha+\beta)} \cong \bigoplus_{s\in\mathbb{Z}} \widehat{\mathcal{V}}_{r+\frac{1}{2}(\lambda+s),-\lambda-s}.$$

Calculation of fusion rules

Teorem

Let $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$, $r_1, r_2, r_3 \in \mathbb{Z}$. *Then:*

$$\dim I \binom{S\Pi_{r_3}(\lambda_3)}{S\Pi_{r_1}(\lambda_1) \quad S\Pi_{r_2}(\lambda_2)} \leq 1.$$

Let us assume there is a non-trivial intertwining operator of type:

$$\binom{S\Pi_{r_3}(\lambda_3)}{S\Pi_{r_1}(\lambda_1) \quad S\Pi_{r_2}(\lambda_2)}$$

in the category of $M \otimes F$*–modules. Then* $\lambda_3 = \lambda_1 + \lambda_2$ *and* $r_3 = r_1 + r_2$*, or* $r_3 = r_1 + r_2 - 1$.

Calculation of fusion rules

Using the following natural isomorphism between spaces of intertwining operators:

$$I_{M\otimes F}\begin{pmatrix} S\Pi_{r_3}(\lambda_3) \\ S\Pi_{r_1}(\lambda_1) \quad S\Pi_{r_2}(\lambda_2) \end{pmatrix} \cong I_M \begin{pmatrix} \Pi_{r_3}(\lambda_3) \\ \Pi_{r_1}(\lambda_1) \quad \Pi_{r_2}(\lambda_2) \end{pmatrix},$$

previous theorem implies our fusion rules result.

Corollary

Let $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$, $r_1, r_2, r_3 \in \mathbb{Z}$. Then there is a non-trivial intertwining operator of type: $\binom{\Pi_{r_3}(\lambda_3)}{\Pi_{r_1}(\lambda_1) \quad \Pi_{r_2}(\lambda_2)}$ in the category of M–modules if and only if $\lambda_3 = \lambda_1 + \lambda_2$ and $r_3 = r_1 + r_2$ or $r_3 = r_1 + r_2 - 1$.

Now the fusion rules in the category of weight M–modules are as follows:

$$\Pi_{r_1}(\lambda_1) \times \Pi_{r_2}(\lambda_2) = \Pi_{r_1+r_2}(\lambda_1 + \lambda_2) \oplus \Pi_{r_1+r_2-1}(\lambda_1 + \lambda_2).$$

Conjectures and future work

- Consider generalized weight modules such that their weight spaces are all ∞–dimensional
- Extend our work to $\mathfrak{gl}(n|m)$
- Try to include Whittaker modules into the fusion category

Contents

Dong-Mason theorem

Let V be a VOA. An automorphism g on V is a map $V \to V$ such that $g(a_n b) = g(a)_n g(b)$, for all $a, b \in V$, $n \in \mathbb{Z}$ and $g(\omega) = \omega$. Let W be a weak V–module. We define the composition $W \circ g$ as a module whose vertex operator is $Y_{W \circ g}(v, z) = Y(gv, z)$, for all $v \in V$.

Teorem (Dong-Mason (1997))

Let V be a VOA and let (M, Y_M) be an irreducible **ordinary** *V–module. Let g be a V–automorphism of prime order p, such that $M \circ g \not\cong M$. Then M is an irreducible $V^{\langle g \rangle}$–module, where $V^{\langle g \rangle}$ is the orbifold subalgebra of V.*

Our main contribution: expanding this result to the whole category of **weak** VOA-modules.

Main theorem

Let g be an automorphism of finite order n. Let W be a V–module. Let us define:

$$\mathcal{M} = W_0 \oplus W_1 \oplus \cdots \oplus W_{n-1},$$

where $W_i = W \circ g^i, i = 0, 1, \cdots, n-1$.

Lemma (ALPY)

Let W be an irreducible V–module and let $\mathcal{M}$ be defined as above. Let us assume that $(w, \ldots, w)$ is a cyclic vector in $\mathcal{M}$, for every $w \in W$, $w \neq 0$. Then W is an irreducible $V^{\langle g \rangle}$–module, where $V^{\langle g \rangle} = \{v \in V \mid gv = v\}$.

Teorem (ALPY)

Let W be an irreducible V–module and let g be an automorphism of finite order n such that $W \circ g^i \not\cong W$, for $i = 1, \ldots, n-1$. Then W is irreducible as a $V^{\langle g \rangle}$–module.

Case $n = 2$

Let θ be an order 2 automorphism on V. Let:

$$V^{+} = \{v \in V \mid \theta(v) = v\}, \quad V^{-} = \{v \in V \mid \theta(v) = -v\}.$$

Then V^{+} is a vertex subalgebra of V, and V^{-} is a V^{+}-module.

Lemma

Let L_i, $i = 1, \dots, t$, be nonisomorphic irreducible weak V–modulesand let $\mathcal{L} = \oplus_{i=1}^{t} L_i$. Then $(w_1, w_2, \dots, w_t)$ is a cyclic vector in $\mathcal{L}$ for every $w_i \neq 0$, $w_i \in \mathcal{L}$.

Teorem

Let W be an irreducible V–module such that $W \circ \theta \ncong W$. Then W is an irreducible V^{+}– module.

Irreducible orbifold subalgebra modules

Let $\zeta_n = e^{2\pi i/n}$ be a primitive n-th root of unity. Let g_n be an automorphism of M, uniquely determined by the following automorphism of $\widehat{\mathcal{A}}$:

$$a(n) \mapsto \zeta_n a(n), \quad a^*(n) \mapsto \zeta_n^{-1} a^*(n) \quad (n \in \mathbb{Z}).$$

Then g_n is an order n automorphism of M.

Teorem

Let $\Lambda = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq 0$. Then $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible module for the orbifold subalgebra $M^{\mathbb{Z}_n} = M^{\langle g_n \rangle}$, for every $n \geq 1$.

Conjectures and future work

In the future we will study three different possible expansions of our main theorem:

- take a non-abelian group of automorphisms:
  conjecture: possible
- take an infinite cyclic group of automorphisms G:
  proved impossible: we have found a counterexample of Whittaker vectors for $\beta\gamma$ ghost (preprint)
- look at twisted modules
  conjecture: possible

Contents

Introduction

- This section is based on a preprint (cf. [AP]) which is joint work with D. Adamović.
- We continue where we left of in the previous section: we study irreducible Whittaker modules for the invariant subalgebra of the Weyl vertex algebra M in the case of infinite dimensional automorphism group
- It turns out that in this case we can not generalise the main theorem from previous section, that is, we have the following theorem:

Teorem

(1) $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ *is a reducible* $\widehat{\mathfrak{gl}}$*–module.*

(2) $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ *is a reducible* $\mathcal{W}_{1+\infty}$*–module with central charge* $c = -1$*.*

- From statement (2) of the previous theorem we also have that $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is reducible as an M^0–module.

Introduction

- Since $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is irreducible as an $M^{\mathbb{Z}_p}$–module for every $p \geq 1$, it is natural to ask if it is irreducible also as an M^0–module. However, we prove that $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a reducible and indecomposable M^0–module.
- Subalgebra M^0 has two important additional realizations:
  - M^0 is isomorphic to the vertex algebra $\mathcal{W}_{1+\infty}$ with central charge $c = -1$. (cf. [KR96])
  - M^0 is isomorphic to the simple module for the Lie algebra $\widehat{\mathfrak{g}l}$, which is the central extension of Lie algebra of infinite matrices.

Main theorem: lemmas

We define operators $E_{i,j} = a(-i)^*(j)$ and the following Casimir element on $\widehat{\mathfrak{gl}}$:

$$I := \sum_{j\in\mathbb{Z}} E_{j,j}.$$

Lemma

(1) *$I \in End(M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}))$ is well defined.*

(2) *The action of I commutes with the action of $\widehat{\mathfrak{gl}}$ on $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$.*

(3) *The action of I commutes with the action of $\mathcal{W}_{1+\infty}$ na $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$.*

Lemma

(1) *For every $n \in \mathbb{Z}_{\geq 1}$, $I^n \mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a nontrivial Wh. vector in $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$.*

(2) *For every $S \subset \mathbb{C}[I]\mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$, let $\langle S\rangle$ be the submodule generated by all Whittaker vectors in S. Then $\left\langle (I-d)\,\mathbb{C}[I]\,\mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}} \right\rangle$ is a proper submodule of $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$, for every $d \in \mathbb{C}$.*

Main theorem

Finally, we prove the main theorem of this section.

Theorem

$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a reducible $\widehat{\mathfrak{gl}}$–module.

$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a reducible $\mathcal{W}_{1+\infty}$–module with a central charge $c = -1$.

Now, from statement (2) it follows that $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a reducible M^0–module. Therefore, the main theorem of previous chapter can not be extended to infinite cyclic groups.

Conjectures

Let $\mathrm{L}_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be an irreducible quotient of the mofule $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$. Some of the problems we plan to study in the future are:

- Prove that $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a cyclic $\widehat{\mathfrak{gl}}$–module.
- Find a free-field realisation of $L_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$.
- Determine the complete set of Whittaker vectors in $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ such that generate maximal submodules in $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Bibliography (1)

Dražen Adamović, Ching Hung Lam, Veronika Pedić, and Nina Yu.
On irreducibility of modules of Whittaker type for cyclic orbifold vertex algebras.
Journal of Algebra, 539:1–23, 2019.

Drazen Adamović and Veronika Pedić.
Whittaker modules for $\mathcal{W}_{1+\infty}$–algebra at $c = -1$.
in preparation.

Dražen Adamović and Veronika Pedić.
On fusion rules and intertwining operators for the Weyl vertex algebra.
Journal of Mathematical Physics, 60(8):081701, 2019.

Bibliography (2)

Chongyin Dong, Haisheng Li, and Geoffrey Mason.
Modular invariance of trace functions in orbifold theory.
Commun. Math. Phys., 214:1–56, 2000.

Chongying Dong and Geoffrey Mason.
On quantum Galois theory.
Duke Mathematical Journal, 86(2):305–321, 1997.

Igor Frenkel, Yi-Zhi Huang, and James Lepowsky.
On axiomatic approaches to vertex operator algebras and modules.
Memoirs of the American Mathematical Soc., 494, 1993.

Victor Gershevich Kac and Andrey Radul.
Representation theory of the vertex algebra $\mathcal{W}_{1+\infty}$.
Transformation groups, 1(1-2):41–70, 1996.