Non-chiral conformal field theory and its application to vertex algebra

Yuto Moriwaki

Research institute for mathematical sciences, Kyoto university

2021/09/23

Introduction

We will discuss two-dimensional non-chiral conformal field theory, their

- Formulation (arXiv:2007.07327)
- Construction (arxiv:2104.10094)
- Deformation (arXiv:2007.07327)

and applications.

Introduction: Deformation in physics

A deformation provides new theories from a known theory.

$$\text{Known theory} \quad \overset{\textit{deform}}{\rightsquigarrow} \quad \text{new theory}$$

Classical mechanics:

$$m\frac{d^2x}{dt^2} = -kx \quad \dashrightarrow \quad m\frac{d^2x}{dt^2} = -kx + gx^3$$

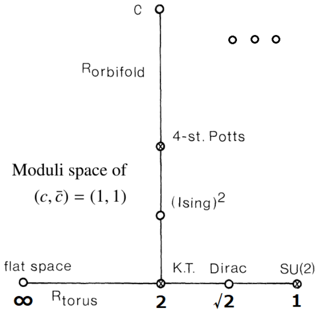
Quantum field theory:

$$\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) \quad \dashrightarrow \quad \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - \frac{g}{4!}\phi^4$$

Deformation in 2d conformal field theory

The figure is the moduli space of 2d CFT with central charge $(1,1)$ conjectured by Dijkgraaf-Verlinde-Verlinde and Ginsparg.

Each point on the line corresponds to a CFT and a line corresponds to a deformation of the CFT.



VOA (chiral part of CFT) cannot deform

$$\text{VOA} \qquad V = \bigoplus_{n\in\mathbb{Z}} V_n, \qquad Y(-,z) : V \to \mathrm{End}\, V[[z^{\pm}]],$$

$$Y(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}.$$

- In physics, the energy changes continuously with deformation.

$$\begin{aligned}\text{energy spectrum} &= L(0)\text{-eigenvalues} \\ &= \text{spin} \in \mathbb{Z} \longrightarrow \text{discrete}.\end{aligned}$$

- Two point correlation function is of the form
  $C \underbrace{(z-w)^n}_{holomorphic}$ with $C \in \mathbb{C}$ and $n \in \mathbb{Z}$.
  - Impossible to deform if we assume that the theory is holomorphic.

Real analytic vertex operator

$$F = \bigoplus_{h,\bar{h}\in\mathbb{R}} F_{h,\bar{h}}, \qquad Y(-,\underline{z}) : F \to \mathrm{End} F[[z^{\mathbb{R}}, \bar{z}^{\mathbb{R}}]],$$
$$Y(a,\underline{z}) = \sum_{h,\bar{h}\in\mathbb{R}} a_{h,\bar{h}} z^{-h-1}\bar{z}^{-\bar{h}-1}.$$

- We have to assume that the vertex operator is single-valued around $z=0$.
  - $z^h\bar{z}^{\bar{h}} = (z\bar{z})^{\bar{h}} z^{h-\bar{h}}$
  - $(z\bar{z})^{\bar{h}} = ||z||^{\bar{h}} = \exp(\bar{h}\log(z\bar{z})) =$ is a single-valued real analytic function on $\mathbb{C}\setminus\{0\}$
- The real analytic vertex operator $Y(-,\underline{z})$ is single-valued around $z=0$ if

$$Y(a,\underline{z}) = \sum_{r\in\mathbb{R}, n\in\mathbb{Z}} a_{r-n,r} ||z||^{-r-1} z^n \in \mathrm{End} F[[z, \bar{z}, ||z||^{\mathbb{R}}]]$$

.

- If we assume
  1. $F_{h_1,\bar{h}_1}(h,\bar{h})F_{h_2,\bar{h}_2} \subset F_{h_1+h_2-h-1,\bar{h}_1+\bar{h}_2-\bar{h}-1}$
  2. $F_{h,\bar{h}} = 0$ if $h-\bar{h} \notin \mathbb{Z}$,

  then $Y(a,\underline{z}) \in \mathrm{End} F[[z,\bar{z},||z||^{\mathbb{R}}]]$ is a consequence.
    - c.f. $V_n(k)V_m \subset V_{n+m-k-1}$ for a VOA V.

$$\mathrm{energy} = L(0)+\overline{L}(0) = h+\bar{h} \in \mathbb{R}$$
$$\mathrm{spin} = L(0)-\overline{L}(0) = h-\bar{h} \in \mathbb{Z},$$

spin is still discrete.

$$C(z-w)^n \underbrace{||z-w||^r}_{\textit{deformable}} \qquad \text{with } n \in \mathbb{Z} \text{ and } r \in \mathbb{R}.$$

Locality

- The vertex operator is assumed to be commutative,
  (Locality) $Y(a,x)Y(b,y) \sim_{a.c.} Y(b,y)Y(a,x)$.
  - This means if we evaluate by $u \in F^\vee = \bigoplus_{h,\bar{h}} F_{h,\bar{h}}^*$ and $v \in F$, then

$$\text{(Correlator)} \qquad u(Y(a,x)Y(b,y)v) =_{a.c.} u(Y(b,y)Y(a,x)v)$$

    both sides converge in $|x| > |y|$ and $|y| > |x|$ respectively and have analytic continuations to the same single-valued functions.
  - Correlators are one of the most important physical quantity.

Goal

Our goal is to provide new mathematical examples of such algebras and construct their deformation.

Summary

We introduce the notion of a full vertex algebra, which is a generalization of a $\mathbb{Z}$-graded vertex algebra.

	vertex algebra	full vertex algebra
vector space	$V = \bigoplus_{n\in\mathbb{Z}} V_n$	$F = \bigoplus_{h,\bar{h}\in\mathbb{R}} F_{h,\bar{h}}$
vertex operator	$\mathrm{End}\, V[[z^{\pm}]]$	$\mathrm{End}\, F[[z,\bar{z},\lvert z\rvert^{\mathbb{R}}]]$
pole	$\mathbb{C}((z))$	$\mathbb{C}((z,\bar{z},\lvert z\rvert^{\mathbb{R}}))$
correlation function	$\mathbb{C}[z^{\pm},(1-z)^{\pm}]$	more complicated (explain later)

- The notion of a full vertex algebra is a reformulation of a full field algebra introduced by Huang and Kong.

Vertex algebras from full vertex algebras

- Let F be a full vertex algebra.
- $D, \bar{D} \in \mathrm{End}\, F$ are defined by $Y(a, \underline{z})\mathbf{1} = a + Daz + \bar{D}a\bar{z} + \dots$

- Then $Y(Da, \underline{z}) = \frac{d}{dz} Y(a, \underline{z})$ and $Y(\bar{D}a, \underline{z}) = \frac{d}{d\bar{z}} Y(a, \underline{z})$.
- $\ker \bar{D} \subset F$ is a vertex algebra and F is a $\ker \bar{D}$-module.
- Moreover, $\ker \bar{D} \otimes \ker D \to F$, a homomorphism, and F is a $\ker \bar{D} \otimes \ker D$-module.

decomposition of CFT

hol CFT $\otimes$ anti-hol CFT $\subset$ 2d CFT in physics,
$\Leftrightarrow$ full vertex algebra homomorphism $t : \ker \bar{D} \otimes \ker D \to F$.

Reduction

A full vertex operator algebra F can be understood through the representation theory of VOAs $\ker \bar{D}$ and $\ker D$ [Moore-Seiberg].
If $\ker \bar{D}$ and $\ker D$ are nice VOAs, then [Huang and Kong]

1. F is decomposed into the direct sum of irreducible $\ker \bar{D} \otimes \ker D$-modules

$$(\textit{Module}) \qquad F = \bigoplus_{i,i' \in \mathrm{Irr} \ker \bar{D} \otimes \ker D} M_i \otimes \overline{M_{i'}}^{n_{ii'}}.$$

2. the vertex operator $Y(-,z)$ on F is also decomposed into the sum of intertwining operators among irreducible modules

$$(\text{intertwining operators}) \qquad Y\binom{k}{ij}(-,z) : M_i \to \mathrm{Hom}(M_j, M_k)[[z^{\mathbb{R}}]].$$

Thus, if we understand $\mathrm{Rep} \ker \bar{D}$ and $\mathrm{Rep} \ker D$ well, then we can construct a full VOA by combining (1) M_i and (2) $Y\binom{k}{ij}(-,z)$.

Known results

Let V be a rational C_2-cofinite VOA + some nice conditions.

- Fuchs, Runkel and Schweigert show that there is a one-to-one correspondence between the full vertex algebras $V \otimes \overline{V} \subset F$ and Frobenius algebra objects in $\mathrm{Rep}\, V \otimes \overline{\mathrm{Rep}\, V}$ (FRS construction).
- Huang and Lepowsky show that $\mathrm{Rep}\, V$ inherits a braided tensor category structure.
- Let $\{M_\lambda\}_{\lambda \in \mathrm{Irr}\, V}$ be the set of the isomorphic classes of the irreducible V-modules. Huang and Kong show that

$$F = \bigoplus_{\lambda \in \mathrm{Irr}\, V} M_\lambda \otimes \overline{M_{\lambda^*}}$$

  inherits a full field algebra (full vertex algebra) structure as an extension of $V \otimes \overline{V}$.
    - The same (dual) modules are used for the holomorphic and anti-holomorphic parts.
    - called a (rational) diagonal model in physics.

Motivation

- Constructions of rational diagonal models are completely done by Huang and Kong.
- However, there are many non-diagonal (indicated by the FRS construction) and irrational CFTs

In fact

A deformation of a rational CFT is almost always an non-diagonal and irrational CFT.

$\longrightarrow$ motivation for introducing the notion of a full vertex algebra.

- Also, it seems to be important to construct algebras explicitly.

Easiest non-trivial example: Ising model

Virasoro vertex operator algebra

- Let us consider the case of

$$\ker \bar{D} = L(\frac{1}{2}, 0) \text{ and } \ker D = \overline{L(\frac{1}{2}, 0)},$$

$L(\frac{1}{2}, 0)$ is the simple Virasoro VOAs of central charge $\frac{1}{2}$ and $\overline{L(\frac{1}{2}, 0)}$ is the "anti-holomorphic" VOA.

  - There are exactly three isomorphism classes of irreducible representations of $L(\frac{1}{2}, 0)$, $L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2}), L(\frac{1}{2}, \frac{1}{16})$.
  - Set $\mathrm{Is} = \{0, \frac{1}{2}, \frac{1}{16}\}$.

- We will explain how to construct a full vertex algebra structure on

$$F_{\mathrm{Ising}} = \bigoplus_{h \in \mathrm{Is}} L(\frac{1}{2}, h) \otimes \overline{L(\frac{1}{2}, h)}.$$

Normalized vertex operator

- Let $h_0, h_1, h_2 \in \mathrm{Is} = \{0, \frac{1}{2}, \frac{1}{16}\}$ and

$$I\binom{h_0}{h_1, h_2}(-, z) : L(\frac{1}{2}, h_1) \to \mathrm{Hom}(L(\frac{1}{2}, h_2), L(\frac{1}{2}, h_0))[z^{\mathbb{R}}]$$

be the normalized intertwining operator such that

$$I\binom{h_0}{h_1, h_2}(\underbrace{|h_1\rangle}_{\text{lowest}}, z)\underbrace{|h_2\rangle}_{\text{lowest}} = \frac{1}{2}\underbrace{|h_0\rangle}_{lowest} z^{h_0 - h_1 - h_2} + \text{higher terms} \dots.$$

if $\{h_0, h_1, h_2\}$ is a permutation of $\{\frac{1}{2}, \frac{1}{16}, \frac{1}{16}\}$

- and by

$$I\binom{h_0}{h_1, h_2}(|h_1\rangle, z)|h_2\rangle = |h_0\rangle z^{h_0 - h_1 - h_2} + \text{higher terms} \dots.$$

otherwise.

Conformal block

- For $h_0, h_1, h_2, h_3, h \in \mathrm{Is}$, set

$$C^h_{h_0,h_1,h_2,h_3}(z_1, z_2) = \langle h_0| \, I\binom{h_0}{h_1\, h}(|h_1\rangle, z_1) I\binom{h}{h_2\, h_3}(|h_2\rangle, z_2)\,|h_3\rangle,$$

  which is called a Virasoro conformal block in physics.
  - $\langle h_0| \in L(\frac{1}{2}, h_0)^*$ is the projection onto the lowest weight vector $|h_0\rangle$.
  - Conformal block is a formal power series with variables z_1 and z_2.
  - The conformal blocks are, in fact, absolutely convergent to multi-valued holomorphic functions in $|z_1| > |z_2|$.
- The Virosoro conformal block of central charge $\frac{1}{2}$ was essentially calculated by Belavin-Polyakov-Zamolodchikov.

Table of conformal block

Table: Conformal blocks [Belavin-Polyakov-Zamolodchikov, M]

(h_0, h_1, h_2, h_3)	h	$C^h_{h_0,h_1,h_2,h_3}(z_1, z_2)$
$(\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16})$	0	$\frac{1}{2}\{z_1 z_2(z_1 - z_2)\}^{-\frac{1}{8}}\left((z_1^{\frac{1}{2}} + z_2^{\frac{1}{2}})^{\frac{1}{2}} + (z_1^{\frac{1}{2}} - z_2^{\frac{1}{2}})^{\frac{1}{2}}\right)$
	$\frac{1}{2}$	$\frac{1}{2}\{z_1 z_2(z_1 - z_2)\}^{-\frac{1}{8}}\left((z_1^{\frac{1}{2}} + z_2^{\frac{1}{2}})^{\frac{1}{2}} - (z_1^{\frac{1}{2}} - z_2^{\frac{1}{2}})^{\frac{1}{2}}\right)$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	0	$\{z_1 z_2(z_1 - z_2)\}^{-1}(z_1^2 - z_1 z_2 + z_2^2)$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{16}, \frac{1}{16})$	0	$z_2^{-\frac{1}{8}}\{z_1(z_1 - z_2)\}^{-\frac{1}{2}}(z_1 - \frac{z_2}{2})$
$(\frac{1}{2}, \frac{1}{16}, \frac{1}{2}, \frac{1}{16})$	$\frac{1}{16}$	$\frac{1}{2}z_1^{-\frac{1}{8}}\{z_2(z_1 - z_2)\}^{-\frac{1}{2}}(z_1 - 2z_2)$
$(\frac{1}{16}, \frac{1}{2}, \frac{1}{2}, \frac{1}{16})$	$\frac{1}{16}$	$\frac{1}{2}(z_1 z_2)^{-\frac{1}{2}}(z_1 - z_2)^{-1}(z_1 + z_2)$
$(\frac{1}{2}, \frac{1}{16}, \frac{1}{16}, \frac{1}{2})$	$\frac{1}{16}$	$\frac{1}{2}\{z_1 z_2\}^{-\frac{1}{2}}(z_1 - z_2)^{-\frac{1}{8}}(z_1 + z_2)$
$(\frac{1}{16}, \frac{1}{2}, \frac{1}{16}, \frac{1}{2})$	$\frac{1}{16}$	$\frac{1}{2}z_1^{-1}\{z_2(z_1 - z_2)\}^{-\frac{1}{2}}(z_1 - 2z_2)$
$(\frac{1}{16}, \frac{1}{16}, \frac{1}{2}, \frac{1}{2})$	0	$z_2^{-1}\{z_1(z_1 - z_2)\}^{-\frac{1}{2}}(z_1 - \frac{z_2}{2})$

Symmetry of conformal block

Important remark

The conformal blocks are almost invariant under the exchange of $h_1 \leftrightarrow h_2$ and $z_1 \leftrightarrow z_2$.

- For example,

$$C^{0}_{\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16}}(z_1,z_2) = z_2^{-\frac{1}{8}}\{z_1(z_1-z_2)\}^{-\frac{1}{2}}(z_1-\frac{z_2}{2})$$
$$C^{0}_{\frac{1}{2},\frac{1}{16},\frac{1}{2},\frac{1}{16}}(z_1,z_2) = z_1^{-\frac{1}{8}}\{z_2(z_1-z_2)\}^{-\frac{1}{2}}(\frac{z_1}{2}-z_2).$$

- We also remark that the conformal blocks are multi-valued holomorphic function on

$$Y_2 = \mathbb{C}^2 \setminus \underbrace{\{z_1=z_2\}\cup\{z_1=0\}\cup\{z_2=0\}}_{\text{Branch point}}$$

How to treat the symmetry of conformal blocks

- The true conformal block is in fact four variables function, $C^h_{h_0,h_1,h_2,h_3}(z_0,z_1,z_2,z_3)$ which is a multi-valued holomorphic function on
$$X_4(\mathbb{C}P^1) = \{(z_0,z_1,z_2,z_3) \in (\mathbb{C}P^1)^4 \mid z_i \neq z_j\}.$$
and convergent in $|z_0| > |z_1| > |z_2| > |z_3|$.
- $C^h_{h_0,h_1,h_2,h_3}(z_1,z_2)$ is obtained by taking $(z_0,z_3) \mapsto (\infty,0)$.

Symmetry of conformal block

Let $\sigma \in S_4$. If you choose a path $\gamma : [0,1] \to X_4(\mathbb{C}P^1)$ from $|z_1| > |z_2| > |z_3| > |z_4|$ to $|z_{\sigma 1}| > |z_{\sigma 2}| > |z_{\sigma 3}| > |z_{\sigma 4}|$. Then, there exists $B^{h,h'}_{h_0,h_1,h_2,h_3}(\gamma) \in \mathbb{C}$ such that

$$A_\gamma C^h_{h_0,h_1,h_2,h_3}(z_0,z_1,z_2,z_3) = \sum_{h' \in \mathrm{Is}} B^{h,h'}_{h_0,h_1,h_2,h_3}(\gamma) C^{h'}_{h_{\sigma 0},h_{\sigma 1},h_{\sigma 2},h_{\sigma 3}}(z_{\sigma 0},z_{\sigma 1},z_{\sigma 2},z_{\sigma 3})$$

Connection matrix

- $B^{h,h'}_{h_0,h_1,h_2,h_3}(\gamma)$ is called a connection matrix which depends only on the homotopy class of γ.
- All we need is $Y(a_1,z_1)Y(a_2,z_2) \sim_{a.c.} Y(a_2,z_2)Y(a_1,z_1)$. Thus, we only consider $C^h_{h_0,h_1,h_2,h_3}(z_1,z_2)$ and let us choose the path $\gamma_0 : [0,1] \to Y_2$ by

$$\gamma_0(t) = (\frac{z_1+z_2}{2} + \exp(\pi i t)\frac{z_1-z_2}{2}, \frac{z_1+z_2}{2} - \exp(\pi i t)\frac{z_1-z_2}{2})$$

  for fixed $|z_1| > |z_2|$.
- We only consider the path γ_0 and omit to write it, $B^{h,h'}_{h_0,h_1,h_2,h_3} = B^{h,h'}_{h_0,h_1,h_2,h_3}(\gamma_0)$, called a Moore-Seiberg data.

Example of connection matrix

- Then, by using the table, we can calculate the connection matrix. For example,

$$
\begin{aligned}
&A_{\gamma_0} C^{0}_{\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16}}(z_1,z_2) \\
&= A_{\gamma_0}\{z_1 z_2(z_1-z_2)\}^{-\frac{1}{8}}\left((z_1^{\frac{1}{2}}+z_2^{\frac{1}{2}})^{\frac{1}{2}}+(z_1^{\frac{1}{2}}-z_2^{\frac{1}{2}})^{\frac{1}{2}}\right) \\
&= \exp(-\frac{1}{8}\pi i)\frac{1+i}{2}\{z_1 z_2(z_1-z_2)\}^{-\frac{1}{8}}\left((z_1^{\frac{1}{2}}+z_2^{\frac{1}{2}})^{\frac{1}{2}}+(z_1^{\frac{1}{2}}-z_2^{\frac{1}{2}})^{\frac{1}{2}}\right) \\
&+ \exp(-\frac{1}{8}\pi i)\frac{1-i}{2}\{z_1 z_2(z_1-z_2)\}^{-\frac{1}{8}}\left((z_1^{\frac{1}{2}}+z_2^{\frac{1}{2}})^{\frac{1}{2}}-(z_1^{\frac{1}{2}}-z_2^{\frac{1}{2}})^{\frac{1}{2}}\right) \\
&= \exp(-\frac{1}{8}\pi i)\frac{1+i}{2} C^{0}_{\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16}}(z_2,z_1) + \exp(-\frac{1}{8}\pi i)\frac{1+i}{2} C^{\frac{1}{2}}_{\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16}}(z_2,z_1)
\end{aligned}
$$

- Thus, $\{B^{h,h'}_{\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16}}\}_{h,h'=0,\frac{1}{2}} = \exp(-\frac{1}{8}\pi i)\begin{pmatrix}\frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2}\end{pmatrix}$

Explicit formula

Proposition 1.1 (M '21)

For any $*$,

$$B_{*,0,*,*} = B_{*,*,0,*} = 1$$

$$B_{*,\frac{1}{2},\frac{1}{2},*} = -1$$

$$B_{a,\frac{1}{2},\frac{1}{16},a'} = B_{a,\frac{1}{16},\frac{1}{2},a'} = \begin{cases} i & (a \text{ or } a' = \frac{1}{2}) \\ -i & \text{otherwise}, \end{cases}$$

$$B^{(b,b')}_{a,\frac{1}{16},\frac{1}{16},a'} = \exp(-\frac{1}{8}\pi i) \times \begin{cases} 1 & (a, a' \neq \frac{1}{16}, a = a') \\ i & (a, a' \neq \frac{1}{16}, a \neq a') \\ \frac{1+i}{2} & (a = a' = \frac{1}{16}, b = b') \\ \frac{1-i}{2} & (a = a' = \frac{1}{16}, b \neq b'). \end{cases}$$

Ising model I

- Let us go back to $F_{\mathrm{Ising}} = \bigoplus_{h \in \mathrm{Is}} L(\frac{1}{2}, h) \otimes \overline{L(\frac{1}{2}, h)}$.
  - Set

$$e_h = |h\rangle \otimes |h\rangle \in L(\frac{1}{2}, h) \otimes \overline{L(\frac{1}{2}, h)}$$

$$I\begin{pmatrix} h_0 \\ h_1\, h_2 \end{pmatrix}(-, \underline{z}) = \underbrace{I\begin{pmatrix} h_0 \\ h_1\, h_2 \end{pmatrix}(-, z)}_{holomorphic} \otimes \underbrace{I\begin{pmatrix} h_0 \\ h_1\, h_2 \end{pmatrix}(-, \bar{z})}_{anti-holomorphic}.$$

- We want to determine, for example,

$$Y(e_{\frac{1}{16}}, z)e_{\frac{1}{16}} = C_{\frac{1}{2}} \cdot I\begin{pmatrix} 0 \\ \frac{1}{16}\, \frac{1}{16} \end{pmatrix}(e_{\frac{1}{16}}, \underline{z})e_{\frac{1}{16}} + C_{\frac{1}{2}} \cdot I\begin{pmatrix} \frac{1}{2} \\ \frac{1}{16}\, \frac{1}{16} \end{pmatrix}(e_{\frac{1}{16}}, z)e_{\frac{1}{16}},$$

where $C_0, C_{\frac{1}{2}} \in \mathbb{C}$ are parameters which need to be determined.

Ising model II

Set

$$A_{\mathrm{Ising}} = \mathbb{C}e_0 \oplus \mathbb{C}e_{\frac{1}{2}} \oplus \mathbb{C}e_{\frac{1}{16}},$$

the lowest weight space.

- From the parameters, $C_0, C_{\frac{1}{2}} \in \mathbb{C}$ etc, we can define a "multiplication" $\cdot : A_{\mathrm{Ising}} \times A_{\mathrm{Ising}} \to A_{\mathrm{Ising}}$. by $e_{\frac{1}{16}} \cdot e_{\frac{1}{16}} = C_0 e_0 + C_{\frac{1}{2}} e_{\frac{1}{2}}$.
- We can show that to satisfy "the locality axiom"

$$e_0 \text{ :unit and } e_{\frac{1}{2}} \cdot e_{\frac{1}{2}} = e_0$$

$$e_{\frac{1}{2}} \cdot e_{\frac{1}{16}} = e_{\frac{1}{16}} = e_{\frac{1}{16}} \cdot e_{\frac{1}{2}}$$

$$e_{\frac{1}{16}} \cdot e_{\frac{1}{16}} = e_0 + e_{\frac{1}{2}}$$

is the unique solution by using the connection matrices $B_{h_0 h_1 h_2 h_3}^{h,h'}$.

Ising model III

- The locality can be checked as for example

$$
\begin{aligned}
& e^*_{\frac{1}{16}} Y(e_{\frac{1}{16}}, z_1) Y(e_{\frac{1}{16}}, z_2) e_{\frac{1}{16}} \\
&= C^0_{\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16}}(z_1,z_2) C^0_{\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16}}(z_1,z_2) + C^{\frac{1}{2}}_{\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16}}(z_1,z_2) C^{\frac{1}{2}}_{\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16}}(z_1,z_2) \\
&=_{a.c.} \frac{1}{2}(z_1\bar{z}_1 z_2\bar{z}_2(z_1-z_2)(\bar{z}_1-\bar{z}_2))^{-\frac{1}{8}} \sqrt{\sqrt{z_1\bar{z}_1} + \sqrt{z_2\bar{z}_2} + \sqrt{(z_1-z_2)(\bar{z}_1-\bar{z}_2)}} \\
&= \frac{1}{2}|z_1 z_2(z_1-z_2)|^{-\frac{1}{4}} \sqrt{|z_1| + |z_2| + |z_1-z_2|} \\
&= e^*_{\frac{1}{16}} Y(e_{\frac{1}{16}}, z_2) Y(e_{\frac{1}{16}}, z_1) e_{\frac{1}{16}}.
\end{aligned}
$$

- To check the locality directly for all vectors is a cumbersome task and we wanted to handle it in a systematic way, which leads us to the notion of a framed algebra.

Remark on locality

- In the case of a vertex algebra V, for any $a, b \in V$, there exits a positive integer N such that $(z-w)^N[Y(a,z), Y(b,w)] = 0$.
- Since

$$e^*_{\frac{1}{16}} Y(e_{\frac{1}{16}}, z_1) Y(e_{\frac{1}{16}}, z_2) e_{\frac{1}{16}}$$
$$= \frac{1}{2}|z_1 z_2(z_1 - z_2)|^{-\frac{1}{4}} \sqrt{|z_1| + |z_2| + |z_1 - z_2|},$$

  there is no such N for a full vertex algebra in general.
- The singularity of a full vertex algebra is more complicated $\sim z^h \bar{z}^{\bar{h}}$.

Framed algebra

Code conformal field theory

Let $(l, r) \in \mathbb{Z}_{\geq 0}$. Hereafter, we assume that

Definition 2.1

A full vertex operator algebra F satisfies

$$\begin{cases} L(\frac{1}{2},0)^{\otimes l} & \subset \ker \bar{D} \\ \overline{L(\frac{1}{2},0)^{\otimes r}} & \subset \ker D \quad \text{as VOAs.} \end{cases}$$

Such a full vertex operator algebra is called an (l, r)-framed full vertex operator algebra.

- We call the corresponding conformal field theory an (l, r)-code conformal field theory.

Framed vertex operator algebra

The notion of $(l, 0)$-framed vertex operator algebra (= means not full) was introduced by Dong-Griess-Höhn and has been studied by many mathematicians. For example, the monster VOA is a $(48, 0)$-framed VOA.

Generalization to the multi-index case

- Recall that $\mathrm{Is} = \{0, \frac{1}{2}, \frac{1}{16}\}$, the set of irreducible modules of $L(\frac{1}{2}, 0)$.
- Set $\mathrm{Is}^{l,r} = \mathrm{Is}^{l} \times \mathrm{Is}^{r}$.
- For $\lambda = (h_1, \dots, h_l, \bar{h}_1, \dots, \bar{h}_r) \in \mathrm{Is}^{l,r}$, set

$$L_{l,r}(\lambda) = \bigotimes_{i=1}^{l} L(\frac{1}{2}, h_i) \otimes \bigotimes_{j=1}^{r} \overline{L(\frac{1}{2}, \bar{h}_j)}.$$

- Then, (l, r)-framed full VOA F is decomposed into

$$F = \bigoplus_{\lambda \in \mathrm{Is}^{l,r}} L_{l,r}(\lambda) \otimes S_\lambda.$$

as a $L_{l,r}(0) = L(\frac{1}{2}, 0)^{\otimes l} \otimes \overline{L(\frac{1}{2}, 0)^{\otimes r}}$-module, where S_λ is the lowest weight space of weight $\lambda \in \mathrm{Is}^{l,r}$.

Framed algebra

- Set $S_F = \bigoplus_{\lambda \in \mathrm{Is}^{l,r}} S_\lambda$, the lowest weight space. Recall that in the case of Ising model $A_{\mathrm{Ising}} = \mathbb{C}e_0 \oplus \mathbb{C}e_{\frac{1}{2}} \oplus \mathbb{C}e_{\frac{1}{16}}$ inherits an algebra structure. Similarly, we have:

Vertex operator to algebra

There exits a unique multiplication $\cdot : S_F \otimes S_F \to S_F$ such that: For $u_1 \otimes a_1 \in L_{l,r}(\lambda^1) \otimes S_{\lambda^1}$ and $u_2 \otimes a_2 \in L_{l,r}(\lambda^2) \otimes S_{\lambda^2}$,

$$Y(u_1 \otimes a_1, \underline{z})u_2 \otimes a_2 = \sum_{\lambda \in \mathrm{Is}^{l,r}} \underbrace{I_{\lambda^1,\lambda^2}^{\lambda}(u_1, z)u_2}_{\text{intertwining operator}} \otimes \underbrace{(a_1 \cdot_\lambda a_2)}_{\text{algebra}-\text{part}}$$

$$\in \bigoplus_{\lambda \in \mathrm{Is}^{l,r}} L_{l,r}(\lambda) \otimes (S_F)_\lambda = F.$$

where $a_1 \cdot_\lambda a_2$ is the projection of the product $a_1 \cdot a_2$ onto λ component $S \to S_\lambda$.

Notation

- For $\lambda^i = (h^i_1, \dots, h^i_l, \bar{h}^i_1, \dots, \bar{h}^i_r) \in \mathrm{Is}^{l,r}$ Set

$$I^{\lambda^0}_{\lambda^1,\lambda^2}(-,z) = \bigotimes_{i=1}^{l} I^{h^0_i}_{h^1_i,h^2_i}(-,z) \otimes \bigotimes_{j=1}^{r} \overline{I}^{h^0_i}_{h^1_i,h^2_i}(-,\bar{z}).$$

- Set $|\lambda\rangle = \bigotimes_{i=1}^{l} |h_i\rangle \otimes \bigotimes_{j=1}^{r} |\bar{h}_j\rangle \in L_{l,r}(\lambda)$, which is the lowest weight vector.

-

$$C^{\lambda}_{\lambda^0,\lambda^1,\lambda^2,\lambda^3}(z_1,z_2) = \langle\lambda^0|\, I\begin{pmatrix}\lambda^0 \\ \lambda^1\lambda\end{pmatrix}(|\lambda^1\rangle, z_1) I\begin{pmatrix}\lambda \\ \lambda^2\lambda^3\end{pmatrix}(|\lambda^2\rangle, z_2)\,|\lambda^3\rangle,$$

  a multi-index conformal block.

-

$$B^{\lambda,\lambda'}_{\lambda^0,\lambda^1,\lambda^2,\lambda^3} = \Pi_{i=1}^{l} B^{h_i,h'_i}_{h^0_i,h^1_i,h^2_i,h^3_i} \Pi_{j=1}^{r} \overline{B^{\bar{h}_j,\bar{h}'_j}_{\bar{h}^0_j,\bar{h}^1_j,\bar{h}^2_j,\bar{h}^3_j}} \in \mathbb{C},$$

  which is a connection matrix for $C^{\lambda}_{\lambda^0,\lambda^1,\lambda^2,\lambda^3}(z_1,z_2)$.

Locality for framed algebra I

Fundamental question

Is it possible to rewrite "the locality",

$$Y(a_1, z_1)Y(a_2, z_2) \sim_{\text{a.c.}} Y(a_2, z_2)Y(a_1, z_1).$$

in terms of the finite-dimensional algebra S_F.

- The product $\cdot$ on S_F should be "commutative".
- Let $\lambda^i \in \mathrm{Is}^{l,r}$ and $a_i \in S_{\lambda^i}$, $a_0^* \in S_{\lambda^0}^*$.Then, by the definition of induced vertex operator, we have:

$$\begin{aligned}
&\langle \lambda^0| \otimes a_0^*, Y_S(|\lambda^1\rangle \otimes a_1, z_1)Y_S(|\lambda^2\rangle \otimes a_2, z_2)\,|\lambda^3\rangle \otimes a_3\rangle \\
&= \sum_{\lambda \in \lambda^2 \star \lambda^3} \langle \lambda^0|\,, I_{\lambda^1\lambda}^{\lambda^0}(|\lambda^1\rangle\,, \underline{z}_1) I_{\lambda^2\lambda^3}^{\lambda}(|\lambda^2\rangle\,, \underline{z}_2)\,|\lambda^3\rangle\rangle \langle a_0^*, a_1 \cdot_{\lambda_0} (a_2 \cdot_\lambda a_3)\rangle \\
&= \sum_{\lambda \in \lambda^2 \star \lambda^3} \underbrace{C_{\lambda^0,\lambda^1,\lambda^2,\lambda^3}^{\lambda}(z_1, z_2)}_{\text{conformal block}} \underbrace{\langle a_0^\vee, a_1 \cdot_{\lambda_0} (a_2 \cdot_\lambda a_3)\rangle}_{\text{algebra-part}}.
\end{aligned}$$

Locality for framed algebra II

- In order to check the locality, it suffices to compare

$$
\begin{aligned}
&A_{\gamma_0} Y_S(-,z_1) Y_S(-,z_2) \\
&\quad \sim A_{\gamma_0} \sum_{\lambda \in \lambda^2 \star \lambda^3} C^{\lambda}_{\lambda^0,\lambda^1,\lambda^2,\lambda^3}(z_1,z_2) \langle a_0^\vee, a_1 \cdot_{\lambda_0} (a_2 \cdot_{\lambda} a_3) \rangle.
\end{aligned}
$$

and

$$
\begin{aligned}
&Y_S(-,z_2) Y_S(-,z_1) \\
&\quad \sim \underbrace{\sum_{\lambda' \in \lambda^1 \star \lambda^3} C^{\lambda}_{\lambda^0,\lambda^2,\lambda^1,\lambda^3}(z_2,z_1) \langle a_0^\vee, a_2 \cdot_{\lambda_0} (a_1 \cdot_{\lambda'} a_3) \rangle}_{\text{1 and 2 are exchanged}}
\end{aligned}
$$

- By using the connection matrix,

Locality for framed algebra III

- The locality is "equivalent" to

$$\sum_{\lambda' \in \lambda^1 \star \lambda^3} C^{\lambda'}_{\lambda^0,\lambda^2,\lambda^1,\lambda^3}(z_2,z_1)\langle a_0^\vee, a_2 \cdot_{\lambda_0} (a_1 \cdot_{\lambda'} a_3)\rangle$$
$$= \sum_{\lambda \in \lambda^2 \star \lambda^3} \langle a_0^\vee, a_1 \cdot_{\lambda_0} (a_2 \cdot_{\lambda} a_3)\rangle \sum_{\lambda'} B^{\lambda,\lambda'}_{\lambda^0,\lambda^1,\lambda^2,\lambda^3} C^{\lambda'}_{\lambda^0,\lambda^2,\lambda^1,\lambda^3}(z_2,z_1).$$

- (By using some argument), we can consider that the conformal blocks $\{C^{\lambda'}_{\lambda^0,\lambda^2,\lambda^1,\lambda^3}(z_2,z_1)\}_{\lambda' \in \mathrm{Is}^{l,r}}$ are linearly independent.
- Hence, the locality is equivalent to

$$\langle a_0^\vee, a_2 \cdot_{\lambda_0} (a_1 \cdot_{\lambda'} a_3)\rangle = \sum_{\lambda \in \lambda^2 \star \lambda^3} B^{\lambda,\lambda'}_{\lambda^0,\lambda^1,\lambda^2,\lambda^3} \langle a_0^\vee, a_1 \cdot_{\lambda_0} (a_2 \cdot_{\lambda} a_3)\rangle$$

Definition of framed algebra

A framed algebra is a finite-dimensional $\mathrm{Is}^{l,r}$-graded vector space $S = \bigoplus_{\lambda \in \mathrm{Is}^{l,r}} S_\lambda$ equipped with a product $\cdot : S \times S \to S$ such that:

FA1) For $\lambda = (h_1, \dots, h_l, \overline{h}_1, \dots, \overline{h}_r) \in \mathrm{Is}^{l,r}$, $S_\lambda = 0$ unless $h_1 + \cdots + h_l - \overline{h}_1 - \cdots - \overline{h}_r \in \mathbb{Z}$ (Spin must be an integer).

FA2) $S_0 = \mathbb{C}\mathbf{1}$ and $\mathbf{1} \cdot v = v \cdot \mathbf{1} = v$ for any $v \in S_F$.

FA3) For any λ^1, λ^2 and $a_1 \in S_{\lambda^1}$, $a_2 \in S_{\lambda^2}$, $a_1 \cdot a_2 \in \bigoplus_{\lambda \in \lambda^1 \star \lambda^2} S_\lambda$.

FA4) For any $\lambda^i \in \mathrm{Is}^{l,r}$ and $a_i \in S_{\lambda^i}$, $a_0^\vee \in S_{\lambda^0}^*$,

$$\langle a_0^\vee, a_2 \cdot_{\lambda_0} (a_1 \cdot_{\lambda'} a_3) \rangle = \sum_{\lambda \in \lambda^2 \star \lambda^3} B_{\lambda^0, \lambda^1, \lambda^2, \lambda^3}^{\lambda, \lambda'} \langle a_0^\vee, a_1 \cdot_{\lambda_0} (a_2 \cdot_\lambda a_3) \rangle$$

New non-associative non-commutative algebra.

Equivalence between categories

Main Theorem 1 (M' 21)

$$\begin{cases} (F, Y(-,z)) & \mapsto (S_F, \cdot) \\ (S, \cdot) & \mapsto (F_S, Y_S(-,z)) \end{cases}$$

give an equivalence between the category of (l,r)-framed full VOA and (l,r)-framed algebra.

Hence, the construction of framed full VOAs is reduced to that of framed algebras.

Construction of framed algebras

Connection matrices I

- To consider the condition (FA4)

$$\langle a_0^\vee, a_2 \cdot_{\lambda_0} (a_1 \cdot_{\lambda'} a_3)\rangle = \sum_{\lambda \in \lambda^2 \star \lambda^3} B^{\lambda,\lambda'}_{\lambda^0,\lambda^1,\lambda^2,\lambda^3} \langle a_0^\vee, a_1 \cdot_{\lambda_0} (a_2 \cdot_{\lambda} a_3)\rangle,$$

  we need to understand the connection matrices $B^{\lambda,\lambda'}_{\lambda^0,\lambda^1,\lambda^2,\lambda^3}$.
- For this purpose, we embed $\mathrm{Is} = \{0, \frac{1}{2}, \frac{1}{16}\}$ into $\mathbb{Z}_2 \times \mathbb{Z}_2$ by

$$0 \leftrightarrow (0,0), \qquad \frac{1}{2} \leftrightarrow (0,1), \qquad \frac{1}{16} \leftrightarrow (1,0).$$

- Then, $\mathrm{Is}^{l,r} = \mathrm{Is}^l \times \mathrm{Is}^r$ is embedded into $\mathbb{Z}_2^{l+r} \times \mathbb{Z}_2^{l+r}$.
- Remark for $(d, c) \in \mathbb{Z}_2^{l+r} \times \mathbb{Z}_2^{l+r}$, d represents the $\frac{1}{16}$-part, and c represents the $\frac{1}{2}$-part.

Connection matrix II

- Define length functions $|\cdot|_l, |\cdot|_r : \mathbb{Z}_2^{l+r} \to \mathbb{Z}_{\geq 0}$ by $|c|_l = \sum_{i=1}^{l} c_i$ and $|c|_r = \sum_{j=1}^{r} \bar{c}_j$ for $c = (c_1, \ldots, c_l, \bar{c}_1, \ldots, \bar{c}_r) \in \mathbb{Z}_2^{l+r}$. Define $|\cdot| : \mathbb{Z}_2^{l+r} \to \mathbb{Z}$ by $|c| = |c|_l - |c|_r$.

Theorem 3.1 (M '21)

$$
\begin{aligned}
&B^{(d^2+d^3,c),(d^1+d^3,c')}_{(d^0,c^0),(d^1,c^1),(d^2,c^2),(d^3,c^3)} \\
&= (-1)^{|c^1c^2|}(-1)^{|d^1c^2(c^0+c^3)|+|d^2c^1(c^0+c^3)|} i^{-|d^1c^2|-|d^2c^1|+|d^1d^2(c^0+c^3)|} \\
&\exp(\frac{-\pi i}{8}|d^1d^2|)(\frac{1+i}{2})^{|d^1d^2d^3|_l}(\frac{1-i}{2})^{|d^1d^2d^3|_r}(-i)^{|d^1d^2d^3(c+c')|}.
\end{aligned}
$$

- $|c| = |c|_l - |c|_r$ appears from the complex conjugate in

$$
B^{\lambda,\lambda'}_{\lambda^0,\lambda^1,\lambda^2,\lambda^3} = \Pi_{i=1}^{l} \underbrace{B^{h_i,h_i'}_{h_i^0,h_i^1,h_i^2,h_i^3}}_{\text{left}} \Pi_{j=1}^{r} \underbrace{\overline{B^{\bar{h}_j,\bar{h}_j'}_{\bar{h}_j^0,\bar{h}_j^1,\bar{h}_j^2,\bar{h}_j^3}}}_{\text{right}}.
$$

Structure code

Let $S = \bigoplus_{(d,c)\in \mathrm{Is}^{l,r}} S_{d,c}$ be an (l,r)-framed algebra.

$$\text{Set} \qquad C_S = \{\alpha \in \mathbb{Z}_2^{l+r} \mid S_{0,\alpha} \neq 0\}$$
$$D_S = \{d \in \mathbb{Z}_2^{l+r} \mid S_{d,c} \neq 0 \text{ for some } c \in \mathbb{Z}_2^{l+r}\}.$$

Proposition 3.2

Assume that the framed algebra S is simple. Then, the following conditions hold:

1. *C_S and D_S are subgroups of $\mathbb{Z}_2^{l+r}$.*
2. *$|\alpha| \in 2\mathbb{Z}$ for any $\alpha \in C_S$ (C_S is an even code).*
3. *$|d| \in 8\mathbb{Z}$ for any $d \in D_S$.*
4. *$D_S \subset C_S^{\perp}$, where $C_S^{\perp} = \{\alpha \in \mathbb{Z}_2^{l+r} \mid |\alpha C_S| \subset 2\mathbb{Z}\}$ (called the dual code).*

- The codes C_S, D_S were introduced by Dong-Griess-Höhn in the case of framed "VOA" and called structure codes.

Construction result

Hereafter, we assume that $l = r$.

- Let $G \subset \mathbb{Z}_2^r$ be a subgroup such that $1^{r+r} = (1, \dots, 1) \in G$.
- Set $D_G = \{(g, g) \in \mathbb{Z}_2^{r+r} \mid g \in G\}$ and $C_G = D_G^\perp \subset \mathbb{Z}_2^{r+r}$, the dual code.

Main Theorem 2

Then, we explicitly construct a simple (r, r)-framed algebra S_G whose structure code is (D_G, C_G).

- S_G gives a frobenius algebra object in $\mathrm{Rep} L_{l,r}(0)$ (FRS construction).
- This result is inspired by Lam-Yamauchi's work for the construction of holomorphic framed VOAs.

Corollary 3.3

$F_G = F_{S_G}$ is a simple full VOA of central charge $(\frac{r}{2}, \frac{r}{2})$.

Dimension formula

- Set $P_G(t) = \sum_{g \in G} t^{|g|} = \sum_{k=0}^{r} \#G[k] t^k$.

Proposition 3.4

$\dim S_G = 2^{2r - \dim G} P_G(\frac{1}{2})$.

- If $r = 1$ and $G = \langle 1 \rangle$, then $P_G(t) = 1 + t$ and $\dim S_G = 2^{2-1}(1 + \frac{1}{2}) = 3$, the Ising model.
- Now, we will classify low rank codes G.
- If $G = G_1 \perp G_2$ as codes, then $S_G \cong S_{G_1} \otimes S_{G_2}$ and hence $F_{S_G} \cong F_{S_{G_1}} \otimes F_{S_{G_2}}$.
- It suffices to consider indecomposable code.

Table of codes and CFT

Table: all indecomposable code CFTs of rank $r \leq 6$

r	code G	current	dim S_G	name
1	$\langle 1 \rangle$	0	3	Ising
2	$\langle 11 \rangle$	$\mathrm{SO}(2)$	10	$R = \sqrt{2}$
3	$\langle 111 \rangle$	$\mathrm{SO}(3)$	36	$\mathrm{SO}(3)_2$
4	$\langle 1111 \rangle$	$\mathrm{SO}(4)$	136	$\mathrm{SO}(4)_1$
	$\langle 1111 \rangle^{\perp}$	0	82	new
5	$\langle 11111 \rangle$	$\mathrm{SO}(5)$	528	$\mathrm{SO}(5)_1$
	$\langle 11000, 00111, 01100 \rangle$	$\mathrm{U}(1)$	276	new
6	$\langle 11111 \rangle$	$\mathrm{SO}(6)$	2080	$\mathrm{SO}(6)_1$
	$\langle 110000, 001111, 101000 \rangle$	$\mathrm{SO}(3)$	1000	new
	$\langle 110000, 001111, 101100 \rangle$	$\mathrm{U}(1)^2$	936	new
	$\langle 110000, 001100, 000011, 101010 \rangle$	0	756	new
	$\langle 111111 \rangle^{\perp}$	0	730	new

Correlators

Since we know the basis and product of S_G explicitly, we have:

Proposition 3.5

Let $\alpha^0, \alpha^1, \alpha^2, \alpha^3 \in C_G^{\mathrm{left}}$. *Then,*

$$\langle Y(e_{\alpha^0}\cdot t_{1^{r+r}}, \underline{z}_0)Y(e_{\alpha^1}\cdot t_{1^{r+r}}, \underline{z}_1)Y(e_{\alpha^2}\cdot t_{1^{r+r}}, \underline{z}_2)Y(e_{\alpha^3}\cdot t_{1^{r+r}}, \underline{z}_3)\mathbf{1}\rangle$$

$$=\delta_{\alpha^0+\alpha^1+\alpha^2+\alpha^3,0}2^{-r}(-1)^{|\alpha^1\alpha^3|}(1, e_{\alpha^0}\cdot e_{\alpha^1}\cdot e_{\alpha^2}\cdot e_{\alpha^3})\Pi_{0\leq i<j\leq 3}\left((z_i-z_j)(\bar{z}_i-\bar{z}_j)\right)^{-\frac{r}{8}}$$

$$\times F(z_0,z_1,z_2,z_3)^{r-|\alpha^0\alpha^1+\alpha^0\alpha^2+\alpha^0\alpha^3+\alpha^1\alpha^2+\alpha^1\alpha^3+\alpha^2\alpha^3|_l}$$

$$G_{01,23}(z_0,z_1,z_2,z_3)^{|\alpha^0\alpha^1+\alpha^2\alpha^3|_l}G_{02,13}(z_0,z_1,z_2,z_3)^{|\alpha^0\alpha^2+\alpha^1\alpha^3|_l}G_{03,12}(z_0,z_1,z_2,z_3)^{|\alpha^0\alpha^3+\alpha^1\alpha^2|}$$

where $G_{ij,kl}(z_1,z_2,z_3,z_4)$ *are defined by*

$$G_{01,23}(z_0,z_1,z_2,z_3) = \left(-|(z_0-z_1)(z_2-z_3)|^{\frac{1}{2}} + |(z_0-z_2)(z_1-z_3)|^{\frac{1}{2}} + |(z_0-z_3)(z_1-z_2)|^{\frac{1}{2}}\right.$$

$$G_{02,13}(z_0,z_1,z_2,z_3) = \left(|(z_0-z_1)(z_2-z_3)|^{\frac{1}{2}} - |(z_0-z_2)(z_1-z_3)|^{\frac{1}{2}} + |(z_0-z_3)(z_1-z_2)|^{\frac{1}{2}}\right)$$

$$G_{03,12}(z_0,z_1,z_2,z_3) = \left(|(z_0-z_1)(z_2-z_3)|^{\frac{1}{2}} + |(z_0-z_2)(z_1-z_3)|^{\frac{1}{2}} - |(z_0-z_3)(z_1-z_2)|^{\frac{1}{2}}\right)$$

Modular invariance

- A $(q, \bar{q})$-character of a full vertex operator algebra $F = \bigoplus_{h,\bar{h}\in\mathbb{R}} F_{h,\bar{h}}$ is defined by

$$\mathrm{Ch}_q F = \sum_{h,\bar{h}\in\mathbb{R}} \dim F_{h,\bar{h}} q^{h-\frac{c}{24}} q^{\bar{h}-\frac{\bar{c}}{24}}.$$

Proposition 3.6

$\mathrm{Ch}_q F_G$ is a real analytic function on upper half plane $\mathcal{H}$ with $q = \exp(2\pi i\tau)$ and it is modular invariant.

Sketch of construction I

1. Let $\mathbb{C}[\hat{C}_G]$ be the twisted group algebra defined by a two-cocycle $\epsilon(-,-)$ on C_G. Then, $\mathbb{C}[\hat{C}_G]$ is a framed algebra with structure code $(0, C_G)$ (No d-part).
2. For $d \in D_G$, define $\mathbb{C}[\hat{C}_G]$-module $A_G(d)$ by $A_G(d) = \mathbb{C}[\hat{C}_G] \otimes_{\mathbb{C}[\Delta^d]} \mathbb{C}t_d$, where $\mathbb{C}t_d$ is a trivial $\mathbb{C}[\Delta^d]$-module.
3. Set $S_G = \bigoplus_{d \in D_G} S_G(d)$.
4. For $d^1, d^2 \in D_G$, define a map $m_{d^1,d^2} : \mathbb{C}[\hat{C}_G] \times \mathbb{C}[\hat{C}_G] \to A_G(d^1 + d^2)$ by

$$m_{d^1,d^2}(e_{\alpha^1}, e_{\alpha^2}) = (-1)^{|d^1 {d_\perp}^2 \alpha^1 \alpha^2| + |d^1 d^2 \alpha^2| + \frac{1}{2}|d^1 \alpha^2|} e_{\alpha^1} \cdot \delta_{d^1 d^2} \cdot e_{\alpha^2} \cdot t_{d^1+d^2}$$

Sketch of proof II

- Then, $m_{d^1,d^2} : \mathbb{C}[\hat{C}_G] \times \mathbb{C}[\hat{C}_G] \to A_G(d^1 + d^2)$ factors through $A_G(d^1) \times A_G(d^2) \to A_G(d^1 + d^2)$.
- Combining m_{d^1,d^2}, we can define $\cdot : S_G \times S_G \to S_G$.
- It suffices to verify (FA4), which can be done by using the explicit formula for the connection matrix $B^{\lambda,\lambda'}_{\lambda^0,\lambda^1,\lambda^2,\lambda^3}$.
- The algebra structure is explicitly!

$$t_{d^1} \cdot t_{d^2} = \sum_{\gamma \in \Delta^{d^1 d^2}} e_\gamma \cdot t_{d^1+d^2}.$$

Deformation of code CFT

Deformation of code conformal field theory

- Deformations of a full vertex algebra are introduced in our previous paper [M '20].
- Set $G^{\perp}[2] = \{g \in G^{\perp} \mid |g| = 2\}$.

Main Theorem 3

Assume that there exists mutually orthogonal vectors $\alpha^1, \dots, \alpha^N \in G^{\perp}[2]$. Then, F_G admits a deformation parametrized by $\mathrm{O}(N,N)/\mathrm{O}(N) \times \mathrm{O}(N)$. Furthermore, for $\sigma \in \mathrm{O}(N,N)$ and $s^0, s^1, s^2, s^3 \in \{\pm\}^N$, the deformed four point function satisfies

$$\langle Y_\sigma(h(s^0)\cdot t_{1^{r+r}}, \underline{z}_0) Y_\sigma(h(s^1)\cdot t_{1^{r+r}}, \underline{z}_1) Y(h(s^1)\cdot t_{1^{r+r}}, \underline{z}_2) Y_\sigma(h(s^3)\cdot t_{1^{r+r}}, \underline{z}_3)\mathbf{1}\rangle$$
$$= 2^{-r+3N} \delta_{p^0+p^1+p^2+p^3,0} F(z_0, z_1, z_2, z_3)^{r-2N}$$
$$\Pi_{0\leq i<j\leq 3} \left((z_i - z_j)(\bar{z}_i - \bar{z}_j)\right)^{\frac{1}{4}\left(p\sigma^{-1}(s^i,s^i), p\sigma^{-1}(s^j,s^j)\right)_l + \frac{1}{4}N - \frac{r}{8}}.$$

- The deformed correlator depends analytically on $\sigma \in O(N,N)$.

Examples of deformations

- For $G = \langle 11 \rangle$, $G^{\perp} = \langle 11 \rangle$. Thus, F_G admits a current-current deformation parametrized by $\mathbb{R} = O(1,1)/O(1) \times O(1)$.
- While for $G = \langle 10, 01 \rangle$, $G^{\perp} = 0$. Thus, F_G does not admit a current-current deformation.
- The number N is equal to the dimension of the Cartan subalgebra of the Lie algebra $(F_G)_{1,0}$.

Many CFTs admit current-current deformations

Table: all indecomposable code CFTs of rank $r \leq 6$

r	code G	current	dim S_G	name
1	$\langle 1 \rangle$	0	3	Ising
2	$\langle 11 \rangle$	$\mathrm{SO}(2)$	10	$R = \sqrt{2}$
3	$\langle 111 \rangle$	$\mathrm{SO}(3)$	36	$\mathrm{SO}(3)_2$
4	$\langle 1111 \rangle$	$\mathrm{SO}(4)$	136	$\mathrm{SO}(4)_1$
	$\langle 1111 \rangle^{\perp}$	0	82	G_4^{even}
5	$\langle 11111 \rangle$	$\mathrm{SO}(5)$	528	$\mathrm{SO}(5)_1$
	$\langle 11000, 00111, 01100 \rangle$	$\mathrm{U}(1)$	276	$G_5^{2;1,1}$
6	$\langle 11111 \rangle$	$\mathrm{SO}(6)$	2080	$\mathrm{SO}(6)_1$
	$\langle 110000, 001111, 101000 \rangle$	$\mathrm{SO}(3)$	1000	$G_6^{2;1,1}$
	$\langle 110000, 001111, 101100 \rangle$	$\mathrm{U}(1)^2$	936	$G_6^{2;1,2}$
	$\langle 110000, 001100, 000011, 101010 \rangle$	0	756	$(E_6^{4;2})^{\perp}$
	$\langle 111111 \rangle^{\perp}$	0	730	G_6^{even}

Code CFTs are ubiquitous

- Code CFT + deformation = large part of the moduli space.
  - Except for $(c, \bar{c}) = (1, 1)$, CFT moduli space is not known well.
- If $\bar{c} = 0$, then a CFT of central charge $(c, \bar{c})$ consists only of holomorphic fields, thus a vertex operator algebra (not full).
  - A VOA does not admit any deformation. Thus, the moduli space is discrete.
- The following table is the classification of chiral CFTs.

central charge	number of CFTs	chiral code CFTs
(8,0)	1	1
(16,0)	2	2
(24,0)	71?	56 [Lam-Shimakura]

Thank you very much