Classification of Holomorphic Vertex Operator <u>Superalgebras</u> of central charge ≤ 24 (j.w. Gerald Höhn, in progress) 1

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<u>Contents</u>:

1) <u>Vertex operator superalgebras (VOSA)</u>

$$V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_{\geq 0}} V_n = \underbrace{\bigoplus_{n\in\mathbb{Z}_{\geq 0}} V_n}_{\substack{V^{\bar 0} \\ \text{(even subVOA)}}} \oplus \underbrace{\bigoplus_{n\in\frac{1}{2}+\mathbb{Z}_{\geq 0}} V_n}_{\substack{V^{\bar 1} \\ \text{(odd part)}}}$$

"correct statistics"

assume non-trivial!

- nice: assume that $V^{\bar 0}$ is rational, C_2-cofinite, CFT-type, simple, self-contr., "pos. cond." (won't write this in the following)
- holomorphic: exactly one irr. module (V itself)

<u>Motivation</u>: superconformal field theories, moonshine for Conway's group, sigma models for K3 surfaces, ...

2) Class. of odd unimodular lattices

$L = L_{ev} \cup L_{odd}$ ← assume non-trivial! odd (= integral but not even), unimodular (L'/L) lattice

(always pos.-def. from now on)

Facts: • $[L : L_{ev}] = 2$

• $|\underbrace{L'_{ev}/L_{ev}}_{=:D}| = 4$ ← discr. form / finite qu. space

• $L_{odd} = L_{ev} + s$, $\quad s + L_{ev} \in D$ with $2(s+L_{ev}) = 0 + L_{ev} \in D$ (order 2), $\quad \frac{\langle s,s \rangle}{2} = \frac{1}{2} \pmod 1$ (quadr. form on D)

• the exact structure of $D := L'_{ev}/L_{ev}$ depends on $rk(L) = rk(L_{ev}) = sign(D) \pmod 8$

e.g. $1 \pmod 8$: 4_1^{+1} $\quad$ ($\mathbb{Z}_4$ with $0, \frac{1}{8}, \frac{1}{2}, \frac{1}{8}$)

$2 \pmod 8$: 2_2^{+2} $\quad$ ($\mathbb{Z}_2 \times \mathbb{Z}_2$ with $0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$)

$\vdots$

$4 \pmod 8$: 2_{II}^{-2} $\quad$ ($\mathbb{Z}_2 \times \mathbb{Z}_2$ with $0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$)

$\vdots$

$\boxed{0 \pmod 8 : 2_{II}^{+2} \quad (\mathbb{Z}_2 \times \mathbb{Z}_2 \text{ with } 0, 0, 0, \tfrac{1}{2})}$ ← 3 possible index-2 extensions (2× even, 1× odd)

Prop: • Let $8 \mid rk(L)$. Then there is a bijection between the odd unimodular lattices L and the edges in the 2-neighbourhood graph of the even unimodular lattices of the same rank.

$$\begin{tikzcd} K_1 \arrow[r, dash, dotted] & L \arrow[r, dash, dotted] & K_2 \\ & L_{ev} \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

edge

• Moreover, for every $m \in \mathbb{Z}_{\geq 0}$, there is a bijection between the odd (or even for $m>0$), unimodular lattices L in rk r with $\vartheta_L = 1 + k q^{\frac{1}{2}} + \ldots$ and in rk $r+m$ with $\vartheta_L = 1 + (k+2m) q^{\frac{1}{2}} + \ldots$

$$\begin{array}{ccc} L & \longmapsto & L \oplus \mathbb{Z}^m \\ rk: r & & r+m \\ n_{1/2}: k & & k+2m \end{array}$$

done for rk 24

Cor: Class. of even, unimod lattices of rank $r \in 8\mathbb{Z}_{\geq 0}$ $\Longrightarrow$ class. of odd, unimod. lattices of rank $\leq r$ (in $\mathbb{Z}_{\geq 0}$)

3) Neighbourhood / orbifold method

$V = V^{\bar 0} \oplus V^{\bar 1}$ (nice) hol. VOSA ($\Rightarrow$ central charge $c \in \frac{1}{2}\mathbb{Z}_{\geq 0}$)

MTC

Facts: • $|\mathrm{Rep}(V^{\bar 0})| = 3$ or 4 (but $\mathrm{glob}(V) = \sum_{W \in \mathrm{Rep}(V)} \mathrm{qdim}(W)^2 = 4$)

conf. weight

• $V^{\bar 1} \in \mathrm{Rep}(V^{\bar 0})$ with $V^{\bar 1} \boxtimes V^{\bar 1} \cong V^{\bar 0}$ (simple current of order 2), $\rho(V^{\bar 1}) = \frac{1}{2} \pmod 1$

• The exact structure of $\mathrm{Rep}(V^{\bar 0})$ depends on $c \pmod 8$

disc. form

(e.g. $\mathrm{Rep}(V^{\bar 0})$ is pointed $\Leftrightarrow |\mathrm{Rep}(V^{\bar 0})| = 4 \Leftrightarrow c \in \mathbb{Z}$, $\rightsquigarrow \mathrm{Rep}(V^{\bar 0}) = \mathcal{C}(D)$)

Virasoro

(Ising model) $c = \frac{1}{2}$: $L_{\frac12}(0), L_{\frac12}(\frac12), L_{\frac12}(\frac{1}{16})$, fusion rules: $\frac12 \times \frac12 = 0$, $\frac12 \times \frac{1}{16} = \frac{1}{16}$, $\frac{1}{16} \times \frac{1}{16} = 0 + \frac12$

(ρ: 0, $\frac12$, $\frac{1}{16}$)

(qdim: 1, 1, $\sqrt{2}$)

$$\boxed{\text{Note: } \Big(\underbrace{L_{\frac12}(0) \oplus L_{\frac12}(\tfrac12)}_{=:F}\Big)^{\otimes 2} \cong bc \cong V_{\mathbb{Z}}}$$

$c = \frac12 \pmod 8$: $\rightsquigarrow$ same MTC

⋮

$$\boxed{c = 0 \pmod 8: \text{ fusion alg. } \mathbb{C}[2_{II}^{+2}] = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2] \text{ with } \rho = 0, 0, 0, \tfrac12 \pmod 1}$$

Thm (Höhn): • Let $8 | c$. Then there is a bijection between the hol. VOSAs and the edges in the 2-neighbourhood (= orbifold) graph of the hol. VOAs

$$\begin{tikzcd} W^{(1)} & V & \tilde{W} \\ & V_{\bar 0} \arrow[ul, bend left, "\mathbb{Z}_2\text{-orbifold}"] \arrow[u] \arrow[ur, bend right, "\mathbb{Z}_2\text{-orbifold}"'] \end{tikzcd}$$

edge

for every $m \in \mathbb{Z}$ (or VOA for $m > 0$)

• Moreover, there is a bijection between the hol. VOSAs of central charge c with $\dim(V_{\frac12}) = k$ and those in central charge $c + m/2$ with $\dim(V_{\frac12}) = k + m$

$$V \longmapsto V \otimes F^{\otimes m}$$

(almost) recently done for $c = 24$

Cor.: Class. of hol. VOAs of central charge $c \in 8\mathbb{Z}_{\geq 0}$

$\Longrightarrow$ class. of hol. VOSAs of central charge $\leq c$ (in $\frac12 \mathbb{Z}_{\geq 0}$)

4) Lattice structure

For any nice VOA W:

mutual commutants (dual pair)

$$W \cong \bigoplus_{(\alpha,\alpha')} n_{\alpha,\alpha'} \, U(\alpha') \otimes V_{\alpha+K}$$

$U_1 = \{0\}$

lattice VOA "orbit lattice"

- W hol. VOA: $\mathrm{Rep}(U) \cong \mathcal{C}(D_U)$ and $D_U \overset{\tau}{\cong} \overline{K'/K}$

  only simple currents

$$W \cong \bigoplus_{\alpha+K \in K'/K} U(\tau(\alpha+K)) \otimes V_{\alpha+K}$$

- W even part of hol. VOSA with $8 \mid c$: 3 cases

  I) $|K'/K| = 4|D_U|$

  II) $|K'/K| = |D_U|$ ⇝ 2 subcases a), b)

  III) $4|K'/K| = |D_U|$

⇝ systematic way to describe extensions of W to { hol VOA / hol. VOSA

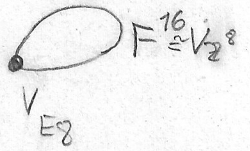
Case	Orb. / Sw. orbifold	Cond. on inner aut(s) $e^{2\pi i h_0}$	Dual pair	
I	inner/inner	$L^h \neq L$	$\lvert K'/K \rvert = 4\lvert D_U \rvert$	edges + loops (within genus)
II a)	inner/inner	$L^h = L$, $\langle h,h\rangle/2 \notin \mathbb{Z}$	$\lvert K'/K \rvert = \lvert D_U \rvert$	mostly loops (or edges with same orbit lattice, e.g. 3)
II b)	inner/non-inner	$L^h = L$, $\langle h,h\rangle/2 \in \mathbb{Z}$		edges between distinct genera
III	non-inner/non-inner	/	$4\lvert K'/K \rvert = \lvert D_U \rvert$	edges + loops (few)

orbifold picture

extension/lattice picture

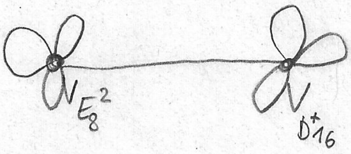
5) Results:

c = 8: (nodes + edges come from lattices)



hol. VOA	hol. VOA	hol VOSA	even part	type	
V_{E_8}	V_{E_8}	$F^{16}\cong V_{\mathbb{Z}^8}$	V_{D_8}	I	→ descends to c=0

c = 16: (nodes come from lattices, but not all edges)



(+ means a certain index 2 extension) →

V_K/U has no weight $\frac{1}{2}$ states

hol. VOA	hol. VOA	hol. VOSA	even part	type	
$V_{D_{16}^+}$	$V_{D_{16}^+}$	$V_K\otimes F^8$	$V_{(D_4 D_{12})^+}$	I	↓c=12
$V_{D_{16}^+}$	$V_{D_{16}^+}$	F^{32}	$V_{D_{16}}$	I	↓c=0
$V_{D_{16}^+}$	$V_{D_{16}^+}$	$V_K\otimes F^2$	$(V_1\cong A_{15})$	I	↓c=15
$V_{D_{16}^+}$	$V_{E_8^2}$	V_K	$V_{(D_8^2)^+}$	I	↓c=16
$V_{E_8^2}$	$V_{E_8^2}$	$V_{E_8}\otimes F^{16}$	$V_{E_8 D_8}$	I	↓c= 8 (or $8\frac{1}{2}$)
$V_{E_8^2}$	$V_{E_8^2}$	$V_{(E_7^2)^+}\otimes F^4$	$V_{(A_1^2E_7^2)^+}$	I	↓c= 14
$V_{E_8^2}$	$V_{E_8^2}$	$U\otimes F$ ↑ $c=15\frac{1}{2}$	$V_{E_8^2}^{S_2}$ $(V_1\cong E_8)$	III	↓c=$15\frac{1}{2}$

c = 24: (neither all nodes nor all edges come from lattices)

↳ 71 nodes = Schellekens' list (assuming the uniqueness of $V^\natural$)

↳ edges: type I : 299 loops + $\frac{414}{2}$ = 207 edges

type IIa : 158 loops + $\frac{18}{2}$ = 9 edges

type IIb : 171 edges

type III : work in progress