# Trialities of $\mathcal{W}$-algebras


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Based on joint work with T. Creutzig

# 1. Notation and examples

Let $\mathfrak{g}$ be a simple, finite-dimensional Lie (super)algebra.

$\mathcal{W}^k(\mathfrak{g}, f)$ the $\mathcal{W}$-algebra at level $k$ associated to $\mathfrak{g}$ and an even nilpotent $f \in \mathfrak{g}$.

It has simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$.

**For this talk**: We will replace $k$ with the **shifted level** $\psi = k + h^\vee$.

$\mathcal{W}^\psi(\mathfrak{g}, f)$ will always denote $\mathcal{W}^k(\mathfrak{g}, f)$ with $k = \psi - h^\vee$.

If $f = f_{\text{prin}}$ is a principal nilpotent, write $\mathcal{W}^\psi(\mathfrak{g}, f) = \mathcal{W}^\psi(\mathfrak{g})$.

**Feigin-Frenkel duality**: $\mathcal{W}^\psi(\mathfrak{g}) \cong \mathcal{W}^{\psi'}({}^L\mathfrak{g})$ where ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra, and $r^\vee \psi \psi' = 1$.

Here $r^\vee$ is the lacing number of $\mathfrak{g}$.

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# 2. Coset construction

Let $\mathcal{V}$ be a VOA and $\mathcal{A} \subseteq \mathcal{V}$ a subVOA

The **coset** $\mathcal{C} = \mathrm{Com}(\mathcal{A}, \mathcal{V})$ is the subVOA of $\mathcal{V}$ which commutes with $\mathcal{A}$, that is, $\mathcal{C} = \{v \in \mathcal{V} |\ [a(z), v(w)] = 0,\ \forall a \in \mathcal{A}\}$.

If $\mathcal{V}$, $\mathcal{A}$ have Virasoro elements $L^{\mathcal{V}}$, $L^{\mathcal{A}}$, then $\mathcal{C}$ has Virasoro element $L^{\mathcal{C}} = L^{\mathcal{V}} - L^{\mathcal{A}}$, and $\mathcal{A} \otimes \mathcal{C} \hookrightarrow \mathcal{V}$ is a conformal embedding.

**Thm**: (Arakawa, Creutzig, L., 2018) Let $\mathfrak{g}$ be simple and simply-laced. We have diagonal embedding

$$V^{\ell+1}(\mathfrak{g}) \hookrightarrow V^{\ell}(\mathfrak{g}) \otimes L_1(\mathfrak{g}), \qquad X^u \mapsto X^u \otimes 1 + 1 \otimes X^u, \qquad u \in \mathfrak{g}.$$

Set

$$\mathcal{C}^{\ell}(\mathfrak{g}) = \mathrm{Com}(V^{\ell+1}(\mathfrak{g}), V^{\ell}(\mathfrak{g}) \otimes L_1(\mathfrak{g})) = (V^{\ell}(\mathfrak{g}) \otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]}.$$

We have an isomorphism of 1-parameter VOAs

$$C^{\ell}(\mathfrak{g}) \cong \mathcal{W}^{\psi}(\mathfrak{g}), \qquad \psi = \frac{\ell + h^{\vee}}{\ell + h^{\vee} + 1}.$$

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# 3. What are trialities of $\mathcal{W}$-algebras?

Let $f \in \mathfrak{g}$ be a nilpotent, and complete $f$ to a copy $\{f, h, e\}$ of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ denote the centralizer of this $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Then $\mathcal{W}^{\psi}(\mathfrak{g}, f)$ has affine subVOA $V^{\psi'}(\mathfrak{a})$, for some level $\psi'$.

By the **affine coset**, we mean $\mathcal{C}^{\psi}(\mathfrak{g}, f) := \text{Com}(V^{\psi'}(\mathfrak{a}), \mathcal{W}^{\psi}(\mathfrak{g}, f))$.

Sometimes we also take invariants under some group of **outer automorphisms**.

**Trialities** are isomorphisms between three different affine cosets

$$\mathcal{C}^{\psi}(\mathfrak{g}, f) \cong \mathcal{C}^{\psi'}(\mathfrak{g}', f') \cong \mathcal{C}^{\psi''}(\mathfrak{g}'', f'').$$

These unify and generalize many well known results, including Feigin-Frenkel duality, coset realization of $\mathcal{W}$-algebras, Feigin-Semikhatov duality, etc.

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# 4. Small hook-type $\mathcal{W}$-algebras in type $A$

**Recall**: For $n \geq 1$, write

$$\mathfrak{sl}_{n+m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \left(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\right) \oplus \left((\mathbb{C}^n)^* \otimes \mathbb{C}^m\right).$$

Let $f_{n,m} \in \mathfrak{sl}_{n+m}$ be the nilpotent which is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

Then $f_{n,m}$ corresponds to the **hook-type partition** $n + 1 + \cdots + 1$.

Define shifted level $\psi = k + n + m$, and define

$$\mathcal{W}^{\psi}(n, m) := \mathcal{W}^{\psi}(\mathfrak{sl}_{n+m}, f_{n+m}),$$

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# 5. Small hook-type $\mathcal{W}$-algebras in type $A$

For $n \geq 1$, $\mathcal{W}^{\psi}(n, m)$ is a common generalization of the following well-known examples.

**Principal**: For $n \geq 2$, $\mathcal{W}^{\psi}(n, 0) = \mathcal{W}^{\psi}(\mathfrak{sl}_n)$

**Subregular**: For $n \geq 2$, $\mathcal{W}^{\psi}(n, 1) = \mathcal{W}^{\psi}(\mathfrak{sl}_{n+1}, f_{\text{subreg}})$

**Trivial**: For $m \geq 1$, $\mathcal{W}^{\psi}(1, m) \cong \mathcal{W}^{\psi}(\mathfrak{sl}_{m+1}, 0) = V^{\psi-m-1}(\mathfrak{sl}_{m+1})$

**Minimal**: For $m \geq 1$, $\mathcal{W}^{\psi}(2, m) \cong \mathcal{W}^{\psi}(\mathfrak{sl}_{m+2}, f_{\text{min}})$.

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# 6. Features of $\mathcal{W}^{\psi}(n, m)$

For $m \geq 2$, $\mathcal{W}^{\psi}(n, m)$ has affine subalgebra

$$V^{\psi-m-1}(\mathfrak{gl}_m) = \mathcal{H} \otimes V^{\psi-m-1}(\mathfrak{sl}_m).$$

Additional **even** generators are in weights $2, 3, \dots, n$ together with $2m$ **even** fields in weight $\frac{n+1}{2}$ which transform under $\mathfrak{gl}_m$ as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$.

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# 6. Features of $\mathcal{W}^{\psi}(n,m)$

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# 7. Small hook-type $\mathcal{W}$-superalgebras of type $A$

For $n + m \geq 2$ and $n \neq m$, write

$$\mathfrak{sl}_{n|m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \left(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\right) \oplus \left((\mathbb{C}^n)^* \otimes \mathbb{C}^m\right).$$

Nilpotent $f_{n|m} \in \mathfrak{sl}_{n|m}$ is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

Define shifted level $\psi = k + n - m$, and let

$$\mathcal{V}^{\psi}(n, m) = \mathcal{W}^{\psi}(\mathfrak{sl}_{n|m}, f_{n|m}),$$

which has level $k = \psi - n + m$.

Case $n = m \geq 2$ slightly different: $\mathcal{V}^{\psi}(n, n) = \mathcal{W}^{\psi}(\mathfrak{psl}_{n|n}, f_{n|n})$.

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For $n \geq 1$, $\mathcal{V}^{\psi}(n, m)$ is a common generalization of the following well-known examples.

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Additional **even** generators in weights $2, 3, \dots, n$, together with $2m$ **odd** fields in weight $\frac{n+1}{2}$ transforming under $\mathfrak{gl}_m$ as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$.

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Consider the affine cosets

$$\mathcal{C}^{\psi}(n,m) = \mathrm{Com}(V^{\psi-m-1}(\mathfrak{gl}_m), \mathcal{W}^{\psi}(n,m)),$$

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**Thm**: (Creutzig-L., 2020) Let $n \geq m$ be non-negative integers. We have isomorphisms of 1-parameter VOAs

$$\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m) \cong \mathcal{D}^{\psi'}(m,n),$$

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$\mathcal{D}^{\psi}(n,0) \cong \mathcal{C}^{\psi^{-1}}(n,0)$ recovers **Feigin-Frenkel duality** in type $A$.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m)$ are of **Feigin-Frenkel type**.

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{D}^{\psi'}(0,n)$ recovers the coset realization of $\mathcal{W}^{\psi}(\mathfrak{sl}_n)$.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{D}^{\psi'}(m,n)$ are of **coset realization type**.

**One more example**:

$$\mathcal{D}^{\psi}(n,1) \cong \mathcal{C}^{\psi^{-1}}(n-1,1) \cong \mathcal{D}^{\psi'}(1,n),$$

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# 12. Sketch of proof, cont'd

**Step 1**: In the $\psi \to \infty$ limit, both $\mathcal{C}^{\psi}(n,m)$ and $\mathcal{D}^{\psi}(n,m)$ become $GL_m$-orbifolds of certain **free field algebras**.

Using **classical invariant theory**, it is shown that

1. $\mathcal{C}^{\psi}(n,m)$ has generating type $\mathcal{W}(2,3,\dots,(m+1)(m+n+1)-1)$,
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**Step 2**: Universal two-parameter $\mathcal{W}_{\infty}$-algebra $\mathcal{W}(c,\lambda)$ serves is a **classifying object** for VOAs of type $\mathcal{W}(2,3,\dots,N)$ for some $N$.

$\mathcal{W}(c,\lambda)$ is freely generated of type $\mathcal{W}(2,3,\dots)$, and is defined over the polynomial ring $\mathbb{C}[c,\lambda]$.

Weight zero component $\mathcal{W}(c,\lambda)[0] \cong \mathbb{C}[c,\lambda]$.

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# 13. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Let $I \subseteq \mathbb{C}[c, \lambda]$ be a prime ideal.

Let $I \cdot \mathcal{W}(c, \lambda)$ be the VOA ideal generated by $I$.

The quotient

$$\mathcal{W}^I(c, \lambda) = \mathcal{W}(c, \lambda)/(I \cdot \mathcal{W}(c, \lambda))$$

is a VOA over $R = \mathbb{C}[c, \lambda]/I$.

$\mathcal{W}^I(c, \lambda)$ is simple for a generic ideal $I$.

But for certain discrete families of ideals $I$, $\mathcal{W}^I(c, \lambda)$ is not simple.

Let $\mathcal{W}_I(c, \lambda)$ be simple graded quotient of $\mathcal{W}^I(c, \lambda)$.

In fact, **all** simple, one-parameter VOAs of type $\mathcal{W}(2, 3, \dots, N)$ satisfying mild hypotheses, are of this form.

Variety $V(I) \subseteq \mathbb{C}^2$ is called the **truncation curve**.

# 13. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Let $I \subseteq \mathbb{C}[c, \lambda]$ be a prime ideal.

Let $I \cdot \mathcal{W}(c, \lambda)$ be the VOA ideal generated by $I$.

The quotient

$$\mathcal{W}^I(c, \lambda) = \mathcal{W}(c, \lambda)/(I \cdot \mathcal{W}(c, \lambda))$$

is a VOA over $R = \mathbb{C}[c, \lambda]/I$.

$\mathcal{W}^I(c, \lambda)$ is simple for a generic ideal $I$.

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Then $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ are of the form $\mathcal{W}_I(c, \lambda)$ for some $I$.

**Step 3**: Explicit truncation curves for $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$.

$\mathcal{W}^{\psi}(n, m)$ is an extension $V^{\psi-m+1}(\mathfrak{gl}_m) \otimes \mathcal{W}_I(c, \lambda)$ for some $I$

Extension is generated by $2m$ fields in weight $\frac{n+1}{2}$ which transform as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$ under $\mathfrak{gl}_m$.

Existence of such an extension uniquely and explicitly determines $I$.

In fact, much more is true: the full OPE algebra of $\mathcal{W}^{\psi}(n, m)$ is uniquely determined from this data.

Same method works for $\mathcal{V}^{\psi}(n, m)$, and triality theorem follows immediately from explicit form of $I$.

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# 15. Some applications

Let $I_{n,m}$ be ideal corresponding to $\mathcal{C}^{\psi}(n,m)$

Nontrivial isomorphisms $\mathcal{C}_{\psi}(n,m) \cong \mathcal{C}_{\psi'}(n',m')$ correspond to intersection points in $V(I_{n,m}) \cap V(I'_{n',m'})$.

All intersections between the curves $V(I_{n,m})$ are rational points.

For example, we can classify isomorphisms $\mathcal{C}_{\psi}(n,m) \cong \mathcal{W}_{\phi}(\mathfrak{sl}_r) = \mathcal{C}_{\phi}(r,0)$.

**Thm**: For all $n \geq 2$, if $r+1$ and $r+n$ are coprime, $\mathcal{W}_{\psi}(n-1,1) = \mathcal{W}_{\psi}(\mathfrak{sl}_n, f_{\text{subreg}})$ is a simple current extension of $V_L \otimes \mathcal{W}_{\phi}(\mathfrak{sl}_r)$, where

$$\psi = \frac{n+r}{n-1}, \qquad \phi = \frac{r+1}{r+n}, \qquad L = \sqrt{nr}\,\mathbb{Z}.$$

Conjectured by Blumenhagen, Eholzer, Honecker, Hornfeck, Hubel (1994).

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Consider the rank $n^2$ $\beta\gamma$-system, $\mathcal{S}(n^2)$, with generators $\beta^{ij}, \gamma^{ij}$ satisfying $\beta^{ij}(z)\gamma^{kl}(w) \sim \delta_{i,k}\delta_{j,l}(z-w)^{-1}$.

It has two commuting actions of $V^{-n}(\mathfrak{sl}_n)$ corresponding to left and right actions of $\mathfrak{sl}_n$ on $n \times n$ matrices.

**Fact**: $\text{Com}(V^{-n}(\mathfrak{sl}_n) \otimes V^{-n}(\mathfrak{sl}_n), \mathcal{S}(n^2)) = \mathcal{S}(n^2)^{\mathfrak{sl}_n[t] \oplus \mathfrak{sl}_n[t]}$ is strongly generated by a Heisenberg field $J$, central fields $\omega_2, \dots, \omega_{n-1}$ of weight $2, 3, \dots, n-1$, and fields of weight $\frac{n}{2}$

$$D^+ = \begin{vmatrix} \beta^{11} & \dots & \beta^{1n} \\ \vdots & & \vdots \\ \beta^{n1} & \dots & \beta^{nn} \end{vmatrix}, \qquad D^- = \begin{vmatrix} \gamma^{11} & \dots & \gamma^{1n} \\ \vdots & & \vdots \\ \gamma^{n1} & \dots & \gamma^{nn} \end{vmatrix},$$

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# 17. Analogue for orthosymplectic types

We define 8 families of $\mathcal{W}$-(super)algebras in a unified framework.

Let $\mathfrak{g}$ be a simple Lie (super)algebra, which will always be either $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, or $\mathfrak{osp}_{n|2r}$.

We have a decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \oplus \rho_{\mathfrak{a}} \otimes \rho_{\mathfrak{b}}$.

Here $\mathfrak{a}$ and $\mathfrak{b}$ are Lie sub(super)algebras of $\mathfrak{g}$, where

1. $\mathfrak{b} = \mathfrak{so}_{2m+1}$ or $\mathfrak{sp}_{2m}$,
2. $\mathfrak{a} = \mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, or $\mathfrak{osp}_{1|2n}$.

$\rho_{\mathfrak{a}}$, $\rho_{\mathfrak{b}}$ transform as the standard representations of $\mathfrak{a}$, $\mathfrak{b}$, respectively.

$\rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{b}}$ can be either even or odd.

# 17. Analogue for orthosymplectic types

We define 8 families of $\mathcal{W}$-(super)algebras in a unified framework.

Let $\mathfrak{g}$ be a simple Lie (super)algebra, which will always be either $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, or $\mathfrak{osp}_{n|2r}$.

We have a decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \oplus \rho_{\mathfrak{a}} \otimes \rho_{\mathfrak{b}}$.

Here $\mathfrak{a}$ and $\mathfrak{b}$ are Lie sub(super)algebras of $\mathfrak{g}$, where

1. $\mathfrak{b} = \mathfrak{so}_{2m+1}$ or $\mathfrak{sp}_{2m}$,
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# 18. Analogue for orthosymplectic types, cont'd

Let $f_{\mathfrak{b}} \in \mathfrak{g}$ be the nilpotent element which is **principal** in $\mathfrak{b}$ and **trivial** in $\mathfrak{a}$.

**Recall**: $\mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{b}})$ has level $\psi - h^{\vee}$.

Let $d_{\mathfrak{a}} = \dim\, \rho_{\mathfrak{a}}$ and $d_{\mathfrak{b}} = \dim\, \rho_{\mathfrak{b}}$.

1. Case 1: $\mathfrak{b} = \mathfrak{so}_{2m+1}$, so $d_{\mathfrak{b}} = 2m+1$.
2. Case 2: $\mathfrak{b} = \mathfrak{sp}_{2m}$, so $d_{\mathfrak{b}} = 2m$.

In both cases, $\mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{b}})$ is of type

$$\mathcal{W}\left(1^{\dim\, \mathfrak{a}}, 2, 4, \dots, 2m, \left(\frac{d_{\mathfrak{b}}+1}{2}\right)^{d_{\mathfrak{a}}}\right).$$

Affine subalgebra is $V^{\psi'}(\mathfrak{a})$ for some level $\psi'$.

The fields in weights $2, 4, \dots, 2m$ are invariant under $\mathfrak{a}$, and the $d_{\mathfrak{a}}$ fields in weight $\frac{d_{\mathfrak{b}}+1}{2}$ transform as the standard $\mathfrak{a}$-module.

# 18. Analogue for orthosymplectic types, cont'd

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# 18. Analogue for orthosymplectic types, cont'd

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# 18. Analogue for orthosymplectic types, cont'd

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# 18. Analogue for orthosymplectic types, cont'd

Let $f_{\mathfrak{b}} \in \mathfrak{g}$ be the nilpotent element which is **principal** in $\mathfrak{b}$ and **trivial** in $\mathfrak{a}$.

**Recall**: $\mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{b}})$ has level $\psi - h^{\vee}$.

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# 19. Analogue for orthosymplectic types, cont'd

For $n, m \geq 0$ we have the following cases where $\mathfrak{b} = \mathfrak{so}_{2m+1}$.

1. **Case 1B**: $\mathfrak{g} = \mathfrak{so}_{2n+2m+2}, \quad \mathfrak{a} = \mathfrak{so}_{2n+1}$.
2. **Case 1C**: $\mathfrak{g} = \mathfrak{osp}_{2m+1|2n}, \quad \mathfrak{a} = \mathfrak{sp}_{2n}$.
3. **Case 1D**: $\mathfrak{g} = \mathfrak{so}_{2n+2m+1}, \quad \mathfrak{a} = \mathfrak{so}_{2n}$.
4. **Case 1O**: $\mathfrak{g} = \mathfrak{osp}_{2m+2|2n}, \quad \mathfrak{a} = \mathfrak{osp}_{1|2n}$.

We write $\mathcal{W}^{\psi}_{1X}(n,m) := \mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{so}_{2m+1}})$, for $X = B, C, D, O$.

**Note**: For $m = 0$, the nilpotent $f_{\mathfrak{b}} \in \mathfrak{g}$ is trivial, so we have

1. $\mathcal{W}^{\psi}_{1B}(n,0) = V^{\psi-2n}(\mathfrak{so}_{2n+2})$,
2. $\mathcal{W}^{\psi}_{1C}(n,0) = V^{\psi+2n+1}(\mathfrak{osp}_{1|2n})$,
3. $\mathcal{W}^{\psi}_{1D}(n,0) = V^{\psi-2n+1}(\mathfrak{so}_{2n+1})$,
4. $\mathcal{W}^{\psi}_{1O}(n,0) = V^{\psi+2n}(\mathfrak{osp}_{2|2n})$.

# 19. Analogue for orthosymplectic types, cont'd

For $n, m \geq 0$ we have the following cases where $\mathfrak{b} = \mathfrak{so}_{2m+1}$.

1. **Case 1B**: $\mathfrak{g} = \mathfrak{so}_{2n+2m+2}, \quad \mathfrak{a} = \mathfrak{so}_{2n+1}$.
2. **Case 1C**: $\mathfrak{g} = \mathfrak{osp}_{2m+1|2n}, \quad \mathfrak{a} = \mathfrak{sp}_{2n}$.
3. **Case 1D**: $\mathfrak{g} = \mathfrak{so}_{2n+2m+1}, \quad \mathfrak{a} = \mathfrak{so}_{2n}$.
4. **Case 1O**: $\mathfrak{g} = \mathfrak{osp}_{2m+2|2n}, \quad \mathfrak{a} = \mathfrak{osp}_{1|2n}$.

We write $\mathcal{W}^{\psi}_{1X}(n, m) := \mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{so}_{2m+1}})$, for $X = B, C, D, O$.

**Note**: For $m = 0$, the nilpotent $f_{\mathfrak{b}} \in \mathfrak{g}$ is trivial, so we have

1. $\mathcal{W}^{\psi}_{1B}(n, 0) = V^{\psi-2n}(\mathfrak{so}_{2n+2})$,
2. $\mathcal{W}^{\psi}_{1C}(n, 0) = V^{\psi+2n+1}(\mathfrak{osp}_{1|2n})$,
3. $\mathcal{W}^{\psi}_{1D}(n, 0) = V^{\psi-2n+1}(\mathfrak{so}_{2n+1})$,
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# 19. Analogue for orthosymplectic types, cont'd

For $n, m \geq 0$ we have the following cases where $\mathfrak{b} = \mathfrak{so}_{2m+1}$.

1. **Case 1B**: $\mathfrak{g} = \mathfrak{so}_{2n+2m+2}$, $\mathfrak{a} = \mathfrak{so}_{2n+1}$.
2. **Case 1C**: $\mathfrak{g} = \mathfrak{osp}_{2m+1|2n}$, $\mathfrak{a} = \mathfrak{sp}_{2n}$.
3. **Case 1D**: $\mathfrak{g} = \mathfrak{so}_{2n+2m+1}$, $\mathfrak{a} = \mathfrak{so}_{2n}$.
4. **Case 1O**: $\mathfrak{g} = \mathfrak{osp}_{2m+2|2n}$, $\mathfrak{a} = \mathfrak{osp}_{1|2n}$.

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**Note**: For $m = 0$, the nilpotent $f_{\mathfrak{b}} \in \mathfrak{g}$ is trivial, so we have

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4. $\mathcal{W}^{\psi}_{1O}(n,0) = V^{\psi+2n}(\mathfrak{osp}_{2|2n})$.

# 20. Analogue for orthosymplectic types, cont'd

For $n \geq 0$ and $m \geq 1$ we have the following cases where $\mathfrak{b} = \mathfrak{sp}_{2m}$.

1. **Case 2B**: $\mathfrak{g} = \mathfrak{osp}_{2n+1|2m}$, $\mathfrak{a} = \mathfrak{so}_{2n+1}$.
2. **Case 2C**: $\mathfrak{g} = \mathfrak{sp}_{2n+2m}$, $\mathfrak{a} = \mathfrak{sp}_{2n}$.
3. **Case 2D**: $\mathfrak{g} = \mathfrak{osp}_{2n|2m}$, $\mathfrak{a} = \mathfrak{so}_{2n}$.
4. **Case 2O**: $\mathfrak{g} = \mathfrak{osp}_{1|2n+2m}$, $\mathfrak{a} = \mathfrak{osp}_{1|2n}$.

We write $\mathcal{W}^{\psi}_{2X}(n,m) := \mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{sp}_{2m}})$, for $X = B, C, D, O$.

For $m = 0$, we define $\mathcal{W}^{\psi}_{2X}(n,0)$ in a different way so that our results hold uniformly for all $n, m \geq 0$.

**Ex**: Let $\mathcal{F}(m)$ and $\mathcal{S}(m)$ denote the rank $m$ free fermion algebra and $\beta\gamma$-system, respectively. Define

$$\mathcal{W}^{\psi}_{2B}(n,0) = \begin{cases} V^{-2\psi-2n+1}(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n+1) & n \geq 1, \\ \mathcal{F}(1) & n = 0. \end{cases}$$

Since $\mathcal{F}(2n+1)$ has an action of $L_1(\mathfrak{so}_{2n+1})$, for $n \geq 1$ $\mathcal{W}^{\psi}_{2B}(n,0)$ has a diagonal action of $V^{-2\psi-2n+2}(\mathfrak{so}_{2n+1})$.

# 20. Analogue for orthosymplectic types, cont'd

For $n \geq 0$ and $m \geq 1$ we have the following cases where $\mathfrak{b} = \mathfrak{sp}_{2m}$.

1. **Case 2B**: $\mathfrak{g} = \mathfrak{osp}_{2n+1|2m}$, $\mathfrak{a} = \mathfrak{so}_{2n+1}$.
2. **Case 2C**: $\mathfrak{g} = \mathfrak{sp}_{2n+2m}$, $\mathfrak{a} = \mathfrak{sp}_{2n}$.
3. **Case 2D**: $\mathfrak{g} = \mathfrak{osp}_{2n|2m}$, $\mathfrak{a} = \mathfrak{so}_{2n}$.
4. **Case 2O**: $\mathfrak{g} = \mathfrak{osp}_{1|2n+2m}$, $\mathfrak{a} = \mathfrak{osp}_{1|2n}$.

We write $\mathcal{W}^{\psi}_{2X}(n,m) := \mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{sp}_{2m}})$, for $X = B, C, D, O$.

For $m = 0$, we define $\mathcal{W}^{\psi}_{2X}(n,0)$ in a different way so that our results hold uniformly for all $n, m \geq 0$.

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# 20. Analogue for orthosymplectic types, cont'd

For $n \geq 0$ and $m \geq 1$ we have the following cases where $\mathfrak{b} = \mathfrak{sp}_{2m}$.

1. **Case 2B**: $\mathfrak{g} = \mathfrak{osp}_{2n+1|2m}$, $\mathfrak{a} = \mathfrak{so}_{2n+1}$.
2. **Case 2C**: $\mathfrak{g} = \mathfrak{sp}_{2n+2m}$, $\mathfrak{a} = \mathfrak{sp}_{2n}$.
3. **Case 2D**: $\mathfrak{g} = \mathfrak{osp}_{2n|2m}$, $\mathfrak{a} = \mathfrak{so}_{2n}$.
4. **Case 2O**: $\mathfrak{g} = \mathfrak{osp}_{1|2n+2m}$, $\mathfrak{a} = \mathfrak{osp}_{1|2n}$.

We write $\mathcal{W}^{\psi}_{2X}(n,m) := \mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{sp}_{2m}})$, for $X = B, C, D, O$.

For $m = 0$, we define $\mathcal{W}^{\psi}_{2X}(n,0)$ in a different way so that our results hold uniformly for all $n, m \geq 0$.

**Ex**: Let $\mathcal{F}(m)$ and $\mathcal{S}(m)$ denote the rank $m$ free fermion algebra and $\beta\gamma$-system, respectively. Define

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Since $\mathcal{F}(2n+1)$ has an action of $L_1(\mathfrak{so}_{2n+1})$, for $n \geq 1$ $\mathcal{W}^{\psi}_{2B}(n,0)$ has a diagonal action of $V^{-2\psi-2n+2}(\mathfrak{so}_{2n+1})$.

# 20. Analogue for orthosymplectic types, cont'd

For $n \geq 0$ and $m \geq 1$ we have the following cases where $\mathfrak{b} = \mathfrak{sp}_{2m}$.

1. **Case 2B**: $\mathfrak{g} = \mathfrak{osp}_{2n+1|2m}$, $\mathfrak{a} = \mathfrak{so}_{2n+1}$.
2. **Case 2C**: $\mathfrak{g} = \mathfrak{sp}_{2n+2m}$, $\mathfrak{a} = \mathfrak{sp}_{2n}$.
3. **Case 2D**: $\mathfrak{g} = \mathfrak{osp}_{2n|2m}$, $\mathfrak{a} = \mathfrak{so}_{2n}$.
4. **Case 2O**: $\mathfrak{g} = \mathfrak{osp}_{1|2n+2m}$, $\mathfrak{a} = \mathfrak{osp}_{1|2n}$.

We write $\mathcal{W}^{\psi}_{2X}(n,m) := \mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{sp}_{2m}})$, for $X = B, C, D, O$.

For $m = 0$, we define $\mathcal{W}^{\psi}_{2X}(n,0)$ in a different way so that our results hold uniformly for all $n, m \geq 0$.

**Ex**: Let $\mathcal{F}(m)$ and $\mathcal{S}(m)$ denote the rank $m$ free fermion algebra and $\beta\gamma$-system, respectively. Define

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Since $\mathcal{F}(2n+1)$ has an action of $L_1(\mathfrak{so}_{2n+1})$, for $n \geq 1$ $\mathcal{W}^{\psi}_{2B}(n,0)$ has a diagonal action of $V^{-2\psi-2n+2}(\mathfrak{so}_{2n+1})$.

# 21. The affine cosets $\mathcal{C}^{\psi}_{iX}(n,m)$

Consider following affine cosets.

1. $\mathcal{C}^{\psi}_{1B}(n,m) := \mathrm{Com}(V^{\psi-2n}(\mathfrak{so}_{2n+1}), \mathcal{W}^{\psi}_{1B}(n,m))^{\mathbb{Z}/2\mathbb{Z}}$,
2. $\mathcal{C}^{\psi}_{1C}(n,m) := \mathrm{Com}(V^{-\psi/2-n-1/2}(\mathfrak{sp}_{2n}), \mathcal{W}^{\psi}_{1C}(n,m))$,
3. $\mathcal{C}^{\psi}_{1D}(n,m) := \mathrm{Com}(V^{\psi-2n+1}(\mathfrak{so}_{2n}), \mathcal{W}^{\psi}_{1D}(n,m))^{\mathbb{Z}/2\mathbb{Z}}$,
4. $\mathcal{C}^{\psi}_{1O}(n,m) := \mathrm{Com}(V^{-\psi/2-n}(\mathfrak{osp}_{1|2n}), \mathcal{W}^{\psi}_{1O}(n,m))^{\mathbb{Z}/2\mathbb{Z}}$,
5. $\mathcal{C}^{\psi}_{2B}(n,m) := \mathrm{Com}(V^{-2\psi-2n+2}(\mathfrak{so}_{2n+1}), \mathcal{W}^{\psi}_{2B}(n,m))^{\mathbb{Z}/2\mathbb{Z}}$,
6. $\mathcal{C}^{\psi}_{2C}(n,m) := \mathrm{Com}(V^{\psi-n-3/2}(\mathfrak{sp}_{2n}), \mathcal{W}^{\psi}_{2C}(n,m))$,
7. $\mathcal{C}^{\psi}_{2D}(n,m) := \mathrm{Com}(V^{-2\psi-2n+3}(\mathfrak{so}_{2n}), \mathcal{W}^{\psi}_{2D}(n,m))^{\mathbb{Z}/2\mathbb{Z}}$,
8. $\mathcal{C}^{\psi}_{2O}(n,m) := \mathrm{Com}(V^{\psi-n-1}(\mathfrak{osp}_{1|2n}), \mathcal{W}^{\psi}_{2O}(n,m))^{\mathbb{Z}/2\mathbb{Z}}$.

# 22. Trialities of orthosymplectic types

**Thm**: (Creutzig-L, 2021) For all integers $m \geq n \geq 0$, we have the following isomorphisms of one-parameter vertex algebras.

$$\mathcal{C}^{\psi}_{2B}(n,m) \cong \mathcal{C}^{\psi'}_{2O}(n,m-n) \cong \mathcal{C}^{\psi''}_{2B}(m,n), \quad \psi' = \frac{1}{4\psi}, \quad \frac{1}{\psi} + \frac{1}{\psi''} = 2,$$

$$\mathcal{C}^{\psi}_{1C}(n,m) \cong \mathcal{C}^{\psi'}_{2C}(n,m-n) \cong \mathcal{C}^{\psi''}_{1C}(m,n), \quad \psi' = \frac{1}{2\psi}, \quad \frac{1}{\psi} + \frac{1}{\psi''} = 1,$$

$$\mathcal{C}^{\psi}_{2D}(n,m) \cong \mathcal{C}^{\psi'}_{1D}(n,m-n) \cong \mathcal{C}^{\psi''}_{1O}(m,n-1), \quad \psi' = \frac{1}{2\psi}, \quad \frac{1}{2\psi} + \frac{1}{\psi''} = 1,$$

$$\mathcal{C}^{\psi}_{1O}(n,m) \cong \mathcal{C}^{\psi'}_{1B}(n,m-n) \cong \mathcal{C}^{\psi''}_{2D}(m+1,n), \quad \psi' = \frac{1}{\psi}, \quad \frac{1}{\psi} + \frac{1}{2\psi''} = 1.$$

# 22. Trialities of orthosymplectic types

**Thm**: (Creutzig-L, 2021) For all integers $m \geq n \geq 0$, we have the following isomorphisms of one-parameter vertex algebras.

$$\mathcal{C}^{\psi}_{2B}(n,m) \cong \mathcal{C}^{\psi'}_{2O}(n,m-n) \cong \mathcal{C}^{\psi''}_{2B}(m,n), \quad \psi' = \frac{1}{4\psi}, \quad \frac{1}{\psi} + \frac{1}{\psi''} = 2,$$

$$\mathcal{C}^{\psi}_{1C}(n,m) \cong \mathcal{C}^{\psi'}_{2C}(n,m-n) \cong \mathcal{C}^{\psi''}_{1C}(m,n), \quad \psi' = \frac{1}{2\psi}, \quad \frac{1}{\psi} + \frac{1}{\psi''} = 1,$$

$$\mathcal{C}^{\psi}_{2D}(n,m) \cong \mathcal{C}^{\psi'}_{1D}(n,m-n) \cong \mathcal{C}^{\psi''}_{1O}(m,n-1), \quad \psi' = \frac{1}{2\psi}, \quad \frac{1}{2\psi} + \frac{1}{\psi''} = 1,$$

$$\mathcal{C}^{\psi}_{1O}(n,m) \cong \mathcal{C}^{\psi'}_{1B}(n,m-n) \cong \mathcal{C}^{\psi''}_{2D}(m+1,n), \quad \psi' = \frac{1}{\psi}, \quad \frac{1}{\psi} + \frac{1}{2\psi''} = 1.$$

# 23. **Special cases**

The isomorphism

$$\mathcal{C}^{\psi}_{1C}(0,m) \cong \mathcal{C}^{\psi'}_{2C}(0,m), \qquad \psi' = \frac{1}{2\psi},$$

is just **Feigin-Frenkel** duality in types $B$ and $C$, since

$$\mathcal{C}^{\psi}_{1C}(0,m) = \mathcal{W}^{\psi}(\mathfrak{so}_{2m+1}), \qquad \mathcal{C}^{\psi'}_{2C}(0,m) \cong \mathcal{W}^{\psi'}(\mathfrak{sp}_{2m}).$$

The isomorphism

$$\mathcal{C}^{\psi}_{2D}(0,m) \cong \mathcal{C}^{\psi'}_{1D}(0,m), \qquad \psi' = \frac{1}{2\psi},$$

is again **Feigin-Frenkel duality** in types $B$ and $C$, since

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# 24. Special cases

The isomorphism

$$\mathcal{C}^{\psi}_{1O}(0,m) \cong \mathcal{C}^{\psi'}_{1B}(0,m), \qquad \psi' = \frac{1}{\psi},$$

is just the $\mathbb{Z}_2$-invariant part of **Feigin-Frenkel duality** in type $D$, since

$$\mathcal{C}^{\psi}_{1O}(0,m) = \mathcal{W}^{\psi}(\mathfrak{so}_{2m+2})^{\mathbb{Z}_2}, \quad \mathcal{C}^{\psi'}_{1B}(0,m) = \mathcal{W}^{\psi'}(\mathfrak{so}_{2m+2})^{\mathbb{Z}_2}.$$

The isomorphism

$$\mathcal{C}^{\psi}_{2B}(0,m) \cong \mathcal{C}^{\psi'}_{2O}(0,m), \qquad \psi' = \frac{1}{4\psi}$$

is the $\mathbb{Z}_2$-invariant part of **Feigin-Frenkel duality** for $\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2m})$, since

$$\mathcal{C}^{\psi}_{2B}(0,m) = \mathcal{W}^{\psi}(\mathfrak{osp}_{1|2m})^{\mathbb{Z}_2}, \quad \mathcal{C}^{\psi'}_{2O}(0,m) = \mathcal{W}^{\psi'}(\mathfrak{osp}_{1|2m})^{\mathbb{Z}_2}.$$

Both can be extended to the full dualities.

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Both can be extended to the full dualities.

# 25. **Special cases**

The isomorphism

$$\mathcal{C}^{\psi}_{2D}(1,m) \cong \mathcal{C}^{\psi'}_{1D}(1,m-1), \qquad \psi' = \frac{1}{2\psi},$$

recovers the $\mathbb{Z}_2$-invariant part of the **Feigin-Frenkel type duality**

$$\mathrm{Com}(\mathcal{H}, \mathcal{W}^{\psi'}(\mathfrak{so}_{2m+1}, f_{\mathrm{subreg}})) \cong \mathrm{Com}(\mathcal{H}, \mathcal{W}^{\psi}(\mathfrak{osp}_{2|2m})),$$

of Creutzig, Genra, and Nakatsuka, since

$$\mathcal{C}^{\psi}_{2D}(1,m) = \mathrm{Com}(\mathcal{H}, \mathcal{W}^{\psi}(\mathfrak{osp}_{2|2m}))^{\mathbb{Z}_2},$$

and

$$\mathcal{C}^{\psi'}_{1D}(1,m-1) = \mathrm{Com}(\mathcal{H}, \mathcal{W}^{\psi'}(\mathfrak{so}_{2m+1}, f_{\mathrm{subreg}}))^{\mathbb{Z}_2}.$$

This can be extended to the full duality.

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This can be extended to the full duality.

# 26. **Special cases**

The isomorphism

$$\mathcal{C}^{\psi}_{1O}(0, n-1) \cong \mathcal{C}^{\psi''}_{2D}(n, 0), \qquad \frac{1}{\psi} + \frac{1}{2\psi''} = 1,$$

recovers the $\mathbb{Z}_2$-invariant part of the **coset realization** of principal $\mathcal{W}$-algebras of type $D$, since

$$\begin{aligned}\mathcal{C}^{\psi''}_{2D}(n, 0) &\cong \mathrm{Com}(V^{-2\psi''-2n+3}(\mathfrak{so}_{2n}), V^{-2\psi''-2n+2}(\mathfrak{so}_{2n}) \otimes \mathcal{F}(2n))^{\mathbb{Z}_2}\\ &\cong \mathrm{Com}(V^{-2\psi''-2n+3}(\mathfrak{so}_{2n}), V^{-2\psi''-2n+2}(\mathfrak{so}_{2n}) \otimes L_1(\mathfrak{so}_{2n}))^{\mathbb{Z}_2},\end{aligned}$$

and

$$\mathcal{C}^{\psi}_{1O}(0, n-1) \cong \mathcal{W}^{\psi}(\mathfrak{so}_{2n})^{\mathbb{Z}_2}.$$

# 27. Special cases

The isomorphism

$$\mathcal{C}^{\psi}_{1C}(n,0) \cong \mathcal{C}^{\psi''}_{1C}(0,n), \qquad \frac{1}{\psi} + \frac{1}{\psi''} = 1$$

yields a **coset realization** of type $B$ and $C$ principal $\mathcal{W}$-algebras.

We have

$$\mathcal{C}^{\psi}_{1C}(n,0) = \mathrm{Com}(V^{-\psi/2-n-1/2}(\mathfrak{sp}_{2n}), V^{\psi+2n+1}(\mathfrak{osp}_{1|2n})),$$
$$\mathcal{C}^{\psi''}_{1C}(0,n) = \mathcal{W}^{\psi''}(\mathfrak{so}_{2n+1}),$$

**Note**: We are using the convention that the form on $\mathfrak{osp}_{1|2n}$ is normalized so that

$$V^{-k/2}(\mathfrak{sp}_{2n}) \hookrightarrow V^{k}(\mathfrak{osp}_{1|2n}).$$

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# 28. Special cases

Suppose we instead use the normalization such that

$$V^k(\mathfrak{sp}_{2n}) \hookrightarrow V^k(\mathfrak{osp}_{1|2n}).$$

We then have

$$\mathcal{C}^{\psi}_{1C}(n,0) = \mathrm{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{osp}_{1|2n})), \qquad k = -\frac{1}{2}(\psi + 2n + 1).$$

This yields the following result.

**Thm**: For all $n \geq 1$, we have the following isomorphism of one-parameter vertex algebras

$$\mathrm{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{osp}_{1|2n})) \cong \mathcal{W}^{\phi}(\mathfrak{so}_{2n+1}), \qquad \phi = \frac{2k+2n+1}{2k+2n+2}.$$

This was also proven in a different way by Creutzig and Genra.

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The isomorphism

$$\mathcal{C}^{\psi}_{2B}(n,0) \cong \mathcal{C}^{\psi''}_{2B}(0,n), \qquad \frac{1}{\psi} + \frac{1}{\psi''} = 2,$$

yields a **coset realization** of $\mathbb{Z}_2$-orbifolds of principal $\mathcal{W}$-algebras of $\mathfrak{osp}_{1|2n}$.

We have

$$\mathcal{C}^{\psi}_{2B}(n,0) \cong \text{Com}(V^{-2\psi-2n+2}(\mathfrak{so}_{2n+1}), V^{-2\psi-2n+1}(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n+1))^{\mathbb{Z}_2},$$

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$$\mathcal{C}^{\psi''}_{2B}(0,n) \cong \mathcal{W}^{\psi''}(\mathfrak{osp}_{1|2n})^{\mathbb{Z}_2}.$$

This can be extended to a coset realization of $\mathcal{W}^{\psi''}(\mathfrak{osp}_{1|2n})$.

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# 30. Special cases

**Thm** (Kac-Wakimoto, 1990) If $k$ is admissible for $\widehat{\mathfrak{so}}_{2n+1}$, we have an embedding

$$L_{k+1}(\mathfrak{so}_{2n+1}) \hookrightarrow L_k(\mathfrak{so}_{2n+1}) \otimes L_1(\mathfrak{so}_{2n_1}).$$

Then simple quotient

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**Recall**: For $\mathfrak{g}$ simply laced, $\mathrm{Com}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))$ is lisse and rational.

We conjecture that this holds for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ as well.

This would imply rationality of corresponding algebras $\mathcal{W}_{\psi''}(\mathfrak{osp}_{1|2n})$.

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