The Conformal Packing Problem and Dual Packing Problem

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The Sphere Packing Problem

The **sphere packing problem** asks for the densest packing of balls of equal radius r without overlap in $\mathbf{R}^n$.

The **density** of a sphere packing is the ratio of the space $\mathbf{R}^n$ covered by the balls. One may assume $r = 1$ by rescaling.
Let Δ_n be the supremum of the density over all possible packings.

The *asymptotic* best known bounds are:

$$-n-1 \;\leq\; \log_2 \Delta_n \;\leq\; -0.5990\, n \quad \text{for } n \longrightarrow \infty.$$

The lower bound is due to *Minkowski* (1905) and holds for all n and provides **lattice packings**, but is not constructive. The upper bound was given by *Kabatiansky & Levenshtein* (1979) and uses linear programming methods.

A related problem is the **kissing number problem**, the packing problem on the sphere. Solved in dimensions 1–4, 8 and 24.

Low Dimensions

An easy method to construct dense packings are **laminated lattices** which are obtained by stacking layers of lower dimensional laminated lattices as dense as possible. One gets:

n	1	2	3	4	5	6	7	8	16	24
L	$\mathbf{Z}$	A_2	A_3	D_4	D_5	D_6	E_7	E_8	BW_{16}	Λ_{24}

They provide the densest lattice packings for at least five dimensions.

$n = 1$ Trivial. Unique.

$n = 2$ Easy. Unique.

$n = 3$ Extremely difficult, computational (*Hales* 1998 & 2014). Not unique.

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$n = 8, 24$ *Cohn & Elkies* (2001) showed that E_8 (root lattice) and Λ_{24} (Leech lattice) are the densest packings up to a factor $(1+\varepsilon)$, $\varepsilon \sim 10^{-8}$, generalizing ideas from the solution of the kissing number problem by *Odlyzko & Sloane* (1979) from S^{n-1} to $\mathbf{R}^n$.

In 2016, it was shown by *Viazovska* ($n = 8$) and *Cohn et al.* ($n = 24$) that these two lattices are indeed the unique solution to the sphere packing problem in those dimensions. Proof uses modular forms for $\Gamma(2)$ in an unexpected way to construct the extremal functional ('magic function').

The Coding Theory Packing Problem

The **binary coding problem** asks for the densest packing of balls of equal radius $d/2$ without overlap in $\mathbf{F}_2^n$. Here, the canonical vector space over the finite field $\mathbf{F}_2$ is equipped with the Hamming metric, the number of different coordinates of two vectors.

The center of the balls form a so called a **binary code** $C \subset \mathbf{F}_2^n$. The code is called **linear** if C is a linear subspace.

The problem is not invariant under scaling of the balls and one considers a function of two variables: Let $A(n,d)$ be the maximal size of any code $C \subset \mathbf{F}_2^n$ with minimal distance d. One compares the information rate

$$R = \frac{\log_2 A(n,d)}{n}$$

with the relative distance d/n, in particular for the limit $n \longrightarrow \infty$.

Analogy: Codes, Lattices and Vertex Operator Algebras.

space	$(\mathbf{F}_2^n, \mathrm{wt})$	$(\mathbf{R}^n, \|\,.\,\|)$	$(V^*_{\mathrm{Vir}_c}, h)$
group	S_n	$O(n)$	$\mathrm{Diff}^+(S^1)$
objects	binary codes	sphere packings	?
integral objects	doubly-even codes C	even lattices L	VOAs V
center density	$1/\|C^\perp/C\|$	$1/\|L^*/L\|$	$1/\dim(\mathcal{T}(V))$

Rational **vertex operator algebras** V of central charge c are an analog of doubly-even binary codes of length n and even lattices of rank n. They are modules for the Virasoro algebra of central charge c which allows to define the **minimal weight** μ_V. The center density is the defined via the **global dimension** $\dim(\mathcal{T}(V))$ of the associated modular tensor category.

Thus we can formulate the **conformal packing problem** in analogy to the packing problems for those codes and lattices.

Center Density 1 – The self-dual case

A binary code C, lattice L or vertex operator algebra V is called **self-dual** if it equals its **dual**, i.e. one has $C = C^{\perp}$, $L = L^*$ or $V = V^*$. By using the associated weight enumerator of the binary code, the theta series of the lattice, or character of the vertex operator algebra, one shows for the minimal weight:

Theorem

- $8|n$ *and* $\mu_C \leq 4\left[\frac{n}{24}\right] + 4$
- $8|n$ *and* $\mu_L \leq 2\left[\frac{n}{24}\right] + 2$
- $8|c$ *and* $\mu_V \leq \left[\frac{n}{24}\right] + 1$ *(H. 1995)*

Extremal Codes, lattices, VOAs are those meeting these bounds. Only known for a few small dimensions, in particular for $n,\ c = 8$: $\mathcal{H}_8$, E_8, $V_{E_8,1}$ and $n,\ c = 24$: $\mathcal{G}_{24}$, Λ_{24}, $V^\natural$.

Problem: No interesting more general VOA packing problem for which $V_{E_8,1}$ and $V^\natural$ is solution.

3-dimensional Quantum Gravity and Black Holes

E. Witten (2007): Self-dual VOAs/CFTs describe 3-dimensional black holes in 3-dimensional quantum gravity via AdS/CFT-correspondence.
Extremal VOAs would correspond to pure quantum gravity.
Extremal character related to Bekenstein bound for Bekenstein-Hawking black hole entropy.

No real reason to restrict to self-dual VOAs/CFTs.
Generalizations to abitrary CFTs considered: *Hellerman* (2009) *Friedan & Keller* (2013), *Mollier, Lin & Yin* (2016), *Hartman, Mazac & Rastelli* (2019): Better and better bounds for 'dimension gap' and other 'spectral invariants' of CFTs using 'modular bootstrap' technique and semidefinite programming.
Although functionals are finally optimal, bounds are usually not sharp.
Problem: Find sharp bounds for the Virasoro spectrum of CFTs.

Our semidefinite programming bounds for $\min(\mu_L, \mu_L^*)$

For a lattice $L \subset \mathbb{R}^n$ let $\mu_L = \min_{v \in L \setminus \{0\}} |v|^2$ and $\mu_L^* = \mu_{L^*}$.

Theorem

For the values of n listed below, the rows provide upper bounds for $\min(\mu_L, \mu_L^*)$ for any lattice L of rank n.

n	5	7	8	16	23	24	32	48
$N = 12$	1.591	1.866	2.00018	3.028	3.892	4.014	4.986	6.932
$N = 24$	1.591	1.866	2.0000005	3.026	3.881	4.00024	4.949	6.802
$N = 36$	1.591	1.866	$2 + 5 \cdot 10^{-9}$	3.026	3.880	4.000007	4.947	6.792

By using the method from the solution of the sphere packing problem in dimension 8 and 24 by Viazovska and Rains et al. I can show $\mu_L^* \leq 2$ resp. $\mu_L^* \leq 4$ in those dimensions.

Theorem

Our bound for $\min(\mu_L, \mu_L^*)$ *is sharp for* $n = 8$ *and* $n = 24$ *where it is reached by the root lattice* E_8 *and the Leech lattice* Λ_{24}.

We restrict now to **even** lattices L. Then $L \subset L^*$ and so $\mu_L^* = \min(\mu_L, \mu_L^*)$. Asymptotically, our upper bound for μ_L^* is slightly better than $n/8 + O(1)$. The general Kabatiansky-Levenshtein bound gives $n/9.795... + O(1)$. Actually one has:

Theorem (Böcherer-Nebe)

For **some** *even lattices* $L \subset \mathbf{R}^n$ *one has* $\mu_L^* \leq n/12 + O(1)$.

Likely, this result can be generalized to **all** even lattices.

Conversely, it is known that there exist even self-dual lattices L which meet the Minkowski bound $\mu_L^* \geq n/17.079... + O(1)$.

A (unitary strongly-rational) **vertex operator algebra** V of central charge $c \in \mathbf{Q}$ is a graded vector space $V = \oplus_{n=0}^{\infty} V_n$, $\dim V_n < \infty$ (together with additional algebraic structure) allowing to define the character

$$\chi_V = q^{-c/24} \sum_{n=0}^{\infty} \dim V_n \; q^n.$$

The V-module $V^* = \bigoplus_{\lambda \in \mathrm{Irr}(V)} M_\lambda$ is called the **dual** of V. Here, the $\{M_\lambda\}_\lambda$ form a system of representatives of irreducible V-modules and their characters $\{\chi_{M_\lambda}\}_\lambda$ define a vector valued modular function for $\mathrm{SL}(2, \mathbf{Z})$ on the upper half-plane if one sets $q = e^{2\pi i \tau}$.

Part of the algebraic structure is the Virasoso algebra $Vir_c = \mathrm{Lie}(\widetilde{\mathrm{Diff}}^+(S^1))$ which allows to decompose V into a direct sum of Vir_c-modules: $V = \bigoplus_{h \leq 0} M(c, h)$.

The smallest occuring $h > 0$ is called the **minimal weight** μ_V of V.

One has

$$\chi_{c,h} := \chi_{M(c,h)} = q^{-c/24+h} \prod_{n=1}^{\infty} (1 - q^n)^{-1} = q^{(-c+1)/24+h} \; \eta(q)^{-1}$$

for $h > 0$ and $\chi_{M(c,0)} = (1 - q) \, \chi_{c,0}$.

Our semidefinite programming bounds for μ_V^*

For a VOA V of central charge c let $\mu_V^* = \min_{v \in \mathcal{P}(V^*)\setminus\{\mathbf{1}\}} \mathrm{wt}(v)$.

Theorem

For the values of c listed below, the rows provide upper bounds for μ_V^* for any VOA of central charge c.

c	$\frac{8}{7}$	4	8	16	$23\frac{1}{2}$	24	32	48
$N = 12$	0.517	0.736	1.0022	1.521	2.005	2.037	2.555	3.603
$N = 24$	0.5165	0.7353	1.000089	1.5082	1.9743	2.0052	2.4978	3.4811
$N = 36$	0.51646	0.73523	1.000008	1.50712	1.97044	2.00107	2.48828	3.4535[illegible]

By adapting the method from the solution of the sphere packing problem in dimension 8 and 24 by Viazovska and Cohn et al. I can show $\mu_V^* \leq 1$ and $\mu_V^* \leq 2$ for vertex operator algebras of central charge 8 and 24, respectively.

Theorem

Our bound for μ_V^* is sharp for $c = 8$ and $c = 24$ where it is reached by the affine Kac Moody VOA for E_8 at level 1 and the moonshine module $V^\natural$.

Asymptotically, our upper bound for μ_V^* is slightly better than $c/16 + O(1)$.

In analogy to codes and lattices, I believe:

Conjecture

For VOAs one has an upper bound $\mu_V^* \leq c/24 + O(1)$.

Conversely, not much is known. To consider general rational VOAs instead of self-dual one alone seems not to help.

I don't know how to construct a (rational) VOA with $\mu_V > 12$.

Proposition

Let V be a VOA of central charge $c > 1$ and let

$$\begin{aligned} F_V &= \frac{1}{2}\left(\frac{\tau}{i}\right)^{1/4} \eta(\tau)\,(\chi_V + \chi_V|S) \\ &= c_0 \left(\frac{\tau}{i}\right)^{1/4} (1-q)\, q^{(1-c)/24} + \sum_{h>0} c_h \left(\frac{\tau}{i}\right)^{1/4} q^{(1-c)/24+h} \end{aligned}$$

where $\eta(\tau) = q^{-1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta-function. Then one has $F_V = F_V|S$ and F_V has the given expansion with non-negative coefficients c_h and $c_0 > 0$. If V has minimum dual conformal weight μ_V^* then $c_h = 0$ for all h in the interval $(0, \mu_V^*)$.

This follows from the decomposition of V and its modules into Virasoro modules, the modular invariance of η, Zhu's theorem about $\chi_V|S$, and positivity of the quantum dimensions of $\mathcal{T}(V)$.

For real t, $\overline{F}_V(t) = F_V(it)$ is real analytic with $\overline{F}_V\left(\frac{1}{t}\right) = \overline{F}_V(t)$.
The functional $\mathcal{D}$ defined by

$$\mathcal{D}[\overline{F}_V] = \sum_{m=0}^{N} a_{2m+1}\,(t\,\partial_t)^{2m+1}(F_V)\,\Big|_{t=1}$$

therefore annihilates $\overline{F}_V$.
For the terms in the expansion of $\overline{F}_V$ with $h > 0$ one has

$$(t\,\partial_t)^m \left(t^{1/4} e^{2\pi(-(1-c)/24-h)t}\right)\Big|_{t=1} = p_m(h)\, e^{2\pi(-(1-c)/24-h)}$$

with a real polynomial p_m of degree m and so

$$\mathcal{D}\left[t^{1/4} e^{2\pi(-(1-c)/24-h)t}\right] = p(h)\, e^{2\pi((c-1)/24-h)}$$

where $p = \sum_{m=0}^{N} a_{2m+1}\, p_{2m+1}$ is a polynomial of degree $2N+1$.
Similarly, for the first term one gets a sum $r = \sum_{m=0}^{N} a_{2m+1}\, r_{2m+1}$ where the real numbers r_{2m+1} depending only on the charge c.

For real numbers a_{2m+1}, $m = 0, \ldots, N$, such that $r > 0$ and $p(h) \geq 0$ for $h \geq \Delta$ one has

$$0 = \mathcal{D}[\overline{F}_V] = c_0\, r + \sum_{h>0} c_h\, p(h)\, e^{2\pi(-(1-c)/24-h)}.$$

Since $c_0\, r$ is positive and $c_h\, p(h)\, e^{2\pi(-(1-c)/24-h)}$ is non-negative for $h \geq \Delta$, at least one of the c_h for $0 < h < \Delta$ has to be nonzero. Thus we have found an upper estimate $\mu_V^* < \Delta$ which must hold for all vertex operator algebras V.

This polynomial optimization problem can be rewritten as a semidefinite programming problem for which powerful numerical solvers exist. I used the solver SDPB for higher dimensional conformal field theory since it allows directly to use the polynomial formulation of the problem as input.

The ratio

$$\lambda = \frac{\wp\left(\frac{\omega_1+\omega_2}{2}\right) - \wp\left(\frac{\omega_2}{2}\right)}{\wp\left(\frac{\omega_1}{2}\right) - \wp\left(\frac{\omega_2}{2}\right)}$$

induces an $\mathrm{SL}(2,\mathbb{F}_2) \cong S_3$ equivariant biholomorphic map

$$\mathbb{H}/\Gamma(2) \longrightarrow \widehat{\mathbb{C}} \setminus \{0,\, 1,\, \infty\}.$$

Let $c = 8$ or $c = 24$. With explicitly given rational functions $Q_c(z)$ and $R_c(z)$, we define the functional

$$\mathcal{C}_c[\varphi] = \int_{1/2}^{1} \varphi(z)\, Q_c(z)\, dz + \frac{1}{2} \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} \varphi(z)\, R_c(z)\, dz$$

for $\varphi \in \mathcal{O}(U)$, $U \subset \widehat{\mathbb{C}} \setminus \{0,\, 1,\, \infty\}$ provided the integrals exists.

Theorem

The integral

$$f_c(h) = \mathcal{D}_c \left[\chi_{c,h}(\lambda^{-1}(z)) \middle| (\mathrm{id} - S) \right]$$

exists for all non-negative real numbers h. It defines a real valued function $f_c(h)$ which is non-negative for $h \geq \frac{c}{16} + \frac{1}{2}$ and vanishes for $h = 0$ and $h \geq \frac{c}{16} + \frac{1}{2} + n$, $n = 0, 1, 2, \ldots$.

Proof: A contour deformation argument shows that

$$\mathcal{D}_c\left[\chi_{c,h}(\lambda^{-1}(z))\big|(\mathrm{id}-S)\right] = 2\,\sin^2(h\,\pi)\int_0^1 \chi_{c,h}(\lambda^{-1}(z))\,Q_c(z)\,dz$$

provided $h > \dfrac{c}{16} + \dfrac{1}{2}$.

Applying $\mathcal{D}_c$ to the S-invariant function $F_V(\lambda^{-1}(z))$ gives as in the numerical case:

Theorem

One has the upper bounds $\mu_V^* \leq 1$ and $\mu_V^* \leq 2$ for VOAs of central charge 8 and 24, respectively. If the bound is reached, then the VOA is self-dual and $\chi_V = j^{1/3}$ and $\chi_V = j - 744$, respectively.

Details: arXiv:1909.05745

Thank you!