Geometric properties of sheaves of coinvariants and conformal blocks

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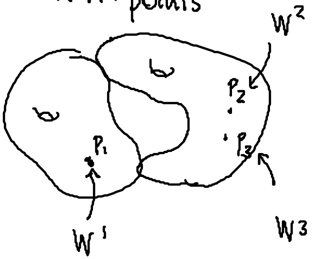
What are we going to see today?

Describe geometric properties of sheaves over $\overline{M}_{g,n}$ attached to V-modules $W^1, \ldots, W^n$ for V a *nice* vertex operator algebra.

based on prop. which hold for V associated with affine Lie Algebras

curves of genus g with n-points



What do I mean by *nice* vertex operator algebras?

$V = \bigoplus_{i \geq 0} V_i$ $\mathbb{N}$
graded vector space

$\mathbf{1} \in V_0$
vacuum vector

$\omega \in V_2$
conformal vector

$Y \colon V \to \mathrm{End}(V)[[z^{\pm 1}]]$
$A \mapsto \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$
state-field correspondence

main

- $V_0 = \mathbf{1}\mathbb{C}$
- $\dim(V_i) < \infty$
- $\{\omega_{(n)}, c_V \mathrm{Id}_V\} \cong \mathrm{Vir}$ c_V central charge $\in \mathbb{Q}$
- V is C_2 cofinite
- V is rational
- V is simple and $\underline{V} \cong V'$

What do I mean by V-modules?

$$W = \bigoplus_{i \geq 0} W_i \quad \mathbb{N}$$

graded vector space

$$Y^W : V \to \mathrm{End}(W)[[z^{\pm 1}]]$$

$$A \mapsto \sum_{n \in \mathbb{Z}} A^W_{(n)} z^{-n-1}$$

state-field correspondence

- $W_0 \neq 0$
- $\dim(W_i) < \infty$
- $\{\omega^W_{(n)}, c_V \mathrm{Id}_W\} \cong \mathrm{Vir}$
- ⋆ $A^W_{(n)} W_j \subseteq W_{j+\deg(A)-n-1}$
- W is simple, hence $L_0(w) = (\deg(w) + \underset{\substack{\uparrow \\ \text{conf. dim}}}{a_W})w \quad \in \mathbb{Q}$

Construction of $\underbrace{\mathbb{V}_{(C,P)}(V;W)}$ SPACE of COINV. $= W \big/ (\mathfrak{g} \otimes A)(W)$

Main idea: $\mathbb{V}_{(C,P)}(V;W)$ is a quotient of W by a Lie algebra encoding the geometric input.

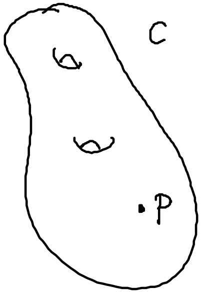
$V = L_\ell(\mathfrak{g})$ $\quad W \in V\text{-mod}$

$\ell \in \mathbb{Z}_{\geq 1}$

$C \setminus P$ affine curve

A Ring of coordinates

$$\begin{array}{ccc} W & & W \\ \circlearrowleft & & \circlearrowleft \\ & & X \otimes f(t_P) \\ & & \uparrow \\ \mathfrak{g} \otimes A & & X \otimes f \end{array}$$



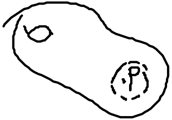
$f(t_P) \in \mathbb{C}((t_P))$

Construction of $\mathbb{V}_{(C,P)}(V;W)$

Main idea: $\mathbb{V}_{(C,P)}(V;W)$ is a quotient of W by a Lie algebra encoding the geometric input.

[A] Construct a Lie algebra $\mathcal{L}_{C\setminus P}(V)$ acting on W.

$$\begin{array}{c} \mathcal{V} \\ \downarrow \\ C \end{array} \qquad \mathcal{V} \xrightarrow{\nabla} \mathcal{V}\otimes\omega_C$$



$D_P^\times \cong \mathbb{C}((t_P))$

$$\boxed{H^0\left(C\setminus P, \frac{\mathcal{V}\otimes\omega_C}{\nabla(\mathcal{V})}\right)}$$

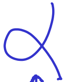
$$\downarrow$$

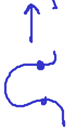
$$H^0\left(D_P^\times, \frac{\mathcal{V}\otimes\omega_C}{\nabla(\mathcal{V})}\right) \underset{t_P}{\cong} \frac{V\otimes\mathbb{C}((t))}{L_{-1}\oplus\frac{d}{dt}} \circlearrowright W$$

C nodal

$\mathcal{V}$ $\quad$ $\mathcal{V}_{\tilde{C}}$

$$\begin{array}{ccc} \mathcal{V} & & \\ \downarrow & & \\ C & \leftarrow & \tilde{C} \end{array}$$ normalization





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[A] Construct a Lie algebra $\mathcal{L}_{C\setminus P}(V)$ acting on W.
[B] Define the *space of coinvariants*

$$\mathbb{V}_{(C,P,t_P)}(V;W) := \frac{W}{\mathcal{L}_{C\setminus P}(V)(W)}.$$

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$$\mathbb{V}_{(C,P,t_P)}(V;W) := \frac{W}{\mathcal{L}_{C\setminus P}(V)(W)}.$$

[C] Forget the coordinate t_P and obtain $\mathbb{V}_{(C,P)}(V;W)$.

Tsuchimoto's paper

$$\begin{tikzcd} \widehat{\mathcal{M}}_{g,1} \arrow[d, "\text{TORSOR } \mathrm{Aut}\,\mathcal{O}"] & C, P, t_P \\ \overline{\mathcal{M}}_{g,1} & C, P \end{tikzcd}$$

Some first properties [D - Gibney - Tarasca]

Under the assumptions that V is nice and $W^1, \dots, W^n$ are simple:

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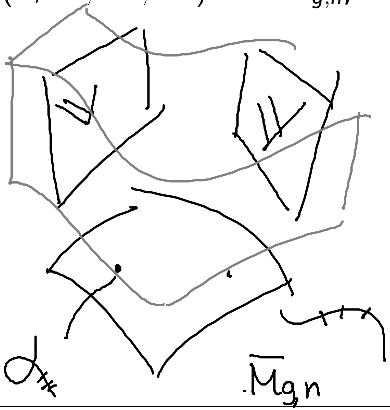

[V.b. over $M_{g,n}$ does not need rationality] only on smooth curves

not need $V = V'$

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  - $\mathbb{V}_{(C,P_\bullet)}(V; W^1, \ldots, W^n)$ are finite dimensional.

~~need~~ C_2

use

$\mathbb{V}_g(W^\bullet)$ coherent sheaf

$\Downarrow$

$\mathbb{V}_g(W^\bullet)^\dagger$ dual coherent sheaf (nice!)

Conformal Blocks

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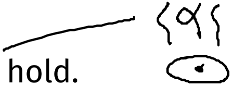
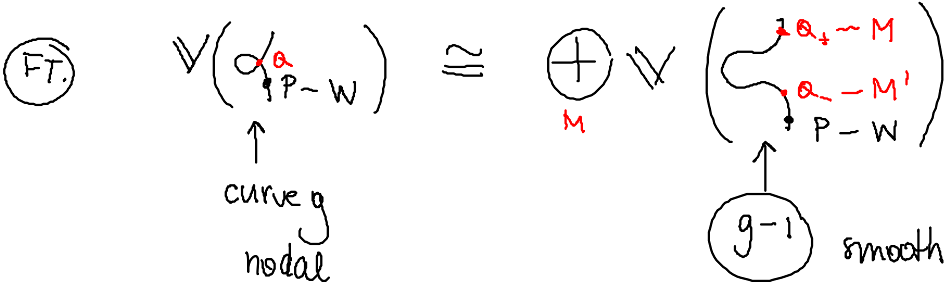
  - Factorization rules and the sewing theorem hold.





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1 - $\mathbb{V}_{(C,P_\bullet)}(V; W^1, \ldots, W^n)$ are finite dimensional.
2 - Factorization rules and the sewing theorem hold.
3 - $\mathbb{V}_g(V; W^1, \ldots, W^n)$ is equipped with a twisted D-module structure with logarithmic singularities along the boundary. Proj. conn.

$\underline{1} + \underline{3} \Rightarrow$ $\mathbb{V}$ is v.b. on $M_{g,n}$

$+ \underline{2}$ extend to boundary

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- The assignment $(g, W^1, \ldots, W^n) \mapsto \mathrm{Ch}(\mathbb{V}_g(V; W^1, \ldots, W^n))$ defines a semisimple CohFT.

Chern character

$\Longrightarrow$ explicitly compute Chern classes of $\mathbb{V}$ knowing

$\boxed{\begin{array}{l} \bullet\ c_V \quad \bullet\ a_W \quad \forall W \\ \bullet\ \mathbb{V}_0(W^1, W^2, W^3) \end{array}}$

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  - modular int. $\mathbb{V}(L_\ell(\mathfrak{g}))^\dagger$ Bun_G
  - global gen over $\overline{M}_{0,n}$

Global generation

Global generation

A vector bundle $\mathbb{V}$ on a variety M is globally generated if there exists a vector space P and a surjective map

$$\underbrace{P \otimes \mathcal{O}_M}_{\text{TRIVIAL BUNDLE}} \twoheadrightarrow \mathbb{V}$$

$\rightsquigarrow$ construct maps from $M \longrightarrow \mathbb{P}^N$ $\quad N = \dim P$

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Theorem [Fakhruddin] The sheaves of coinvariants $\mathbb{V}_0(L_\ell(\mathfrak{g}); W^1, \ldots, W^n)$ are a quotient of $(\underbrace{W_0^1 \otimes \cdots \otimes W_0^n}_{\mathfrak{g}\text{-repr.}}) \otimes \mathcal{O}_{\overline{M}_{0,n}}$, hence they are globally generated.

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Theorem [D - Gibney] The sheaves of coinvariants $\mathbb{V}_0(V; W^1, \ldots, W^n)$ are a quotient of $(W_0^1 \otimes \cdots \otimes W_0^n) \otimes \mathcal{O}_{\overline{M}_{0,n}}$, hence they are globally generated **if V is strongly generated in degree 1**.

Do not need rationality, $V \cong V'$

First example

$V = L(\frac{1}{2}, 0)$ so that $\mathrm{Rep}(V) = \{L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2}), L(\frac{1}{2}, \frac{1}{16})\} = \{V, W(\frac{1}{2}), W(\frac{1}{16})\}$.

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Over $\overline{M}_{0,4}$ there are only three sheaves with degree different from zero:

$\overset{\shortparallel}{\mathbb{P}^1}$

$\forall\ \mathbb{V}$ G-gen. $\Rightarrow$ $\deg \mathbb{V} \geq 0$

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bundle	rank	degree
$\mathbb{V}_0(V; W(\frac{1}{2}), W(\frac{1}{2}), W(\frac{1}{2}), W(\frac{1}{2}))$	1	2

CohFT

$\uparrow$

Atiyah algebra

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$\mathbb{V}_0(V; W(\frac{1}{2}), W(\frac{1}{2}), W(\frac{1}{16}), W(\frac{1}{16}))$	1	1
$\longrightarrow$ $\mathbb{V}_0(V; W(\frac{1}{16}), W(\frac{1}{16}), W(\frac{1}{16}), W(\frac{1}{16}))$	2	-1 $\longleftarrow$

sheaf of coin. cannot be g.gen!

Second example

$V = V_L$ with $L = \mathbb{Z}\epsilon$ lattice with $(\epsilon, \epsilon) = 2 \cdot 4$, so that $\mathrm{Rep}(V_L) \cong \mathbb{Z}/8\mathbb{Z} = \{W_i\}_{i=0}^{7}$.

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W4

$$c_1(\mathbb{V}_0(2,2,2,2)) = \left(\frac{4}{16}\psi_1 + \frac{4}{16}\psi_2 + \frac{4}{16}\psi_3 + \frac{4}{16}\psi_4\right) - \left(\frac{16}{16}\delta_{[2,2][2,2]} + \frac{16}{16}\delta_{[2,2][2,2]} + \frac{16}{16}\delta_{[2,2][2,2]}\right).$$

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$V = V_L$ with $L = \mathbb{Z}\epsilon$ lattice with $(\epsilon, \epsilon) = \overset{2\cdot k}{2 \cdot 4}$, so that $\mathrm{Rep}(V_L) \cong \mathbb{Z}/8\mathbb{Z} = \{W_i\}_{i=0}^{7}$.

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$$c_1(\mathbb{V}_0(2,2,2,2)) = \left(\frac{4}{16}\psi_1 + \frac{4}{16}\psi_2 + \frac{4}{16}\psi_3 + \frac{4}{16}\psi_4\right) - \underbrace{\left(\frac{16}{16}\delta_{[2,2][2,2]} + \frac{16}{16}\delta_{[2,2][2,2]} + \frac{16}{16}\delta_{[2,2][2,2]}\right)}_{\text{boundary!}}.$$

$$\deg(\mathbb{V}_0(2,2,2,2)) = \left(\frac{4}{16} + \frac{4}{16} + \frac{4}{16} + \frac{4}{16}\right) - \left(\frac{16}{16} + \frac{16}{16} + \frac{16}{16}\right) = 1 - 3 = \underset{-k}{-2}.$$

Comments on the proof — Take 1

Theorem [D - Gibney] If V is strongly generated in degree 1, then the sheaves of coinvariants $\mathbb{V}_0(V; W^1, \ldots, W^n)$ are a quotient of $(W_0^1 \otimes \cdots \otimes W_0^n) \otimes \mathcal{O}_{\overline{M}_{0,n}}$, hence they are globally generated.

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1. The map $\underbrace{(W_0^1 \otimes \cdots \otimes W_0^n) \otimes \mathcal{O}_{\overline{M}_{g,n}}}_{\text{TRIVIAL over } \overline{M}_{gn}} \to \mathbb{V}_g(V; W^1, \ldots, W^n)$ is always defined.

with coordinates

$$\begin{array}{c} W_0^1 \otimes \cdots \otimes W_0^n \\ \downarrow \\ \boxed{W^1 \otimes \cdots \otimes W^n} \twoheadrightarrow \mathbb{V}(W^1, \ldots W^n) \end{array}$$

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2. Enough to show that

$$W_0^1 \otimes \cdots \otimes W_0^n \hookrightarrow W^1 \otimes \cdots \otimes W^n \twoheadrightarrow \mathbb{V}_{(C,P_\bullet,t_\bullet)}(V; W^1, \ldots, W^n)$$

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3. We are left to show that: Induction on degree

$$\underbrace{W^1 \otimes \cdots \otimes W^n}_{d} = \boxed{\leq d-1} + \mathcal{L}_{C \setminus P}(V) \cdot W^1 \otimes \cdots \otimes W^n$$

Comments on the proof — Take 2

Theorem [D - Gibney] If V is strongly generated in degree 1, then the sheaves of coinvariants $\mathbb{V}_0(V; W^1, \ldots, W^n)$ are a quotient of $(W^1_0 \otimes \cdots \otimes W^n_0) \otimes \mathcal{O}_{\overline{M}_{0,n}}$, hence they are globally generated.

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A. **strongly generated in degree 1:** we can assume that $w = w^1 \otimes \cdots \otimes w^n$ has one component

$$w^i = A_{(-j)} u^i \qquad\qquad u^i \in W^1_{\deg(w^i)-j} \quad A \in V_1 \text{ and } j \geq 1$$

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B. **g=0:** we can construct an element $\boxed{\sigma \in \mathcal{L}_{C \setminus P_\bullet}(V)}$ such that

- $\sigma_{P_i} = A_{(-j)} +$ lower degree terms
- $\deg(\sigma_{P_k}) \leq \deg(A) - 1 = 0$ for $i \neq k$.

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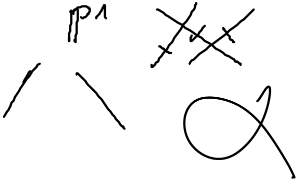

Conclusion

$$\deg\left(w - \sigma(w^1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ w^i}}{u^i} \otimes \cdots w^n)\right) \leq \deg(w) - 1 \qquad \square$$

Concluding comments

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