Lie algebra cohomology

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1 History

Citing [1]:

> In mathematics, Lie algebra cohomology is a cohomology theory for Lie algebras. It was first introduced in 1929 by Élie Cartan to study the topology of Lie groups and homogeneous spaces by relating cohomological methods of Georges de Rham to properties of the Lie algebra. It was later extended by Claude Chevalley and Samuel Eilenberg (1948) to coefficients in an arbitrary Lie module.

2 Objects in Lie theory

We begin and define Lie algebras, Lie groups and enveloping algebras. Their relation is given by: Lie algebras appear as ...

- tangent spaces at 1 of Lie groups and
- by "unenveloping" associative algebras.

We then continue to define representations (ie. modules) of such objects.

2.1 Lie algebras

A *Lie algebra* over a field k[1] is a vector space $\mathfrak{g}$ with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the *Lie bracket*) satisfying *bilinearity*, *alternativity* and the *Jacobi identity*.

One should think of a Lie algebra as a (non-unitary) ring with $[\cdot,\cdot]$ being multiplication, although it need not be associative (or commutative). For example one defines a *derivation* on $\mathfrak{g}$ as a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ satisfying *Leibniz law*:

$$\delta[x, y] = [\delta x, y] + [x, \delta y]. \tag{1}$$

[1] The definition for Lie algebra makes sense for any ring k. However, we will mainly discuss the case $k \in \{\mathbb{R}, \mathbb{C}\}$, namely when $\mathfrak{g}$ is the Lie algebra for a Lie group G (real) or its complexification $\mathfrak{g}_{\mathbb{C}}$.

Also the notions of *homomorphism*, *subalgebra*, *ideal*, *direct sum* and *quotient algebra* extend to Lie algebras. For example a linear map $f : \mathfrak{g} \to \mathfrak{g}'$ between underlying vector spaces of $\mathfrak{g}$ and $\mathfrak{g}'$ is a homomorphism of Lie algebras if it is compatible with multiplication: $f[x, y] = [fx, fy]$.

If $\mathfrak{i}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ *splits* (ie. has a section), then $\mathfrak{g} = \mathfrak{g}/\mathfrak{i} \ltimes \mathfrak{i}$ is called a *semidirect product*. *Levi's theorem* says that every finite dimensional Lie algebra is a semidirect product (by its radical and the complementary (Levi) subalgebra).

The *dimension* of $\mathfrak{g}$ is the cardinality of a minimal generating set of $\mathfrak{g}$[2].

2.2 Lie groups

By a *(real) Lie group G* we mean a group object in the category of finite dimensional real smooth manifolds[3].

2.3 Enveloping algebras

Let A be an *associative algebra* with multiplication $\cdot$. The underyling vector space of A can be equipped with a Lie bracket $[x, y] = x \cdot y - y \cdot x$[4] resulting in a Lie algebra $\mathrm{Lie}(A)$. The algebra A is called an *enveloping algebra* of $\mathrm{Lie}(A)$. This construction is functorial and turns Lie into a (*right adjoint*) functor

$$\mathrm{Lie} : \{k\text{-associative algebras}\} \rightleftarrows \{k\text{-Lie algebras}\} : U. \qquad (2)$$

The left adjoint U assigns to $\mathfrak{g}$ its *universal enveloping algebra* $U(\mathfrak{g})$. $U(\mathfrak{g})$ can be constructed as the quotient $T(\mathfrak{g})/I$ of

- $T(\mathfrak{g}) = k \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \mathfrak{g}^{\otimes 3} \oplus \dots$ (the *tensor algebra* of $\mathfrak{g}$) by
- I the two-sided ideal in $T(\mathfrak{g})$ generated by elements of the form $x \otimes y - y \otimes x - [x, y] \in \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2}$.

In particular, if $\mathfrak{g}$ is abelian, then $U(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g})$ is the *symmetric algebra* of the vector space underlying $\mathfrak{g}$.

Example 2.1. Let M be a vector space. Then $\mathrm{End}(M)$ is an associative algebra and we denote $\mathfrak{gl}(M) := \mathrm{Lie}(\mathrm{End}(M))$. Also, if $k = \mathbb{R}$, then $\mathfrak{gl}(M) = T_1\mathrm{GL}(M)$.

Theorem 2.2. *(The PBW*[5]*-theorem) The natural map* $\mathfrak{g} \to \mathrm{Lie}U(\mathfrak{g})$ *is injective.*

[2] As usual the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ generated by a set $S \subset \mathfrak{g}$ is the smallest subalgebra containing S.

[3] Multiplication and inversion being smooth is equivalent to $(x, y) \mapsto x^{-1}y$ being smooth.

[4] The Jacobi identity follows from associativity of $\cdot$.

[5] Named after Henri Poincaré, Garret Birkhoff and Ernst Witt.

3 Representations

Always $\mathfrak{g}$ denotes a Lie algebra over a field (or ring) k. If we speak about G, it is assumed to be a real Lie group with Lie algebra $\mathfrak{g}$ and in particular $k = \mathbb{R}$ and $\mathfrak{g}$ possibly complexified. By $\mathfrak{k}$ we mean a Lie subalgebra of $\mathfrak{g}$. By K we mean a subgroup of G, mostly assumed compact and maximal. We do simplify notation and always write π for the representation, whether from $\mathfrak{g}$ or G or ...

If M is an k-vector space we can associate to it a Lie algebra $\mathfrak{gl}(M)$ (over k), an associative algebra $\mathrm{End}(M)$ (over k) and if $k \in \{\mathbb{R}, \mathbb{C}\}$ a Lie group $\mathrm{GL}(M)$. Then basically, "morphisms" to $\mathfrak{gl}(M)$ (resp. $\mathrm{End}(M)$, resp. $\mathrm{GL}(M)$) represent *representations on M*.

3.1 $\mathfrak{g}$-modules

A $\mathfrak{g}$-*module* is a pair (M, π) where M is an k-vector space and $\pi : \mathfrak{g} \to \mathfrak{gl}(M)$ in the category of Lie algebras over k.

3.2 $U\mathfrak{g}$-modules

A $U\mathfrak{g}$-module is an k-vector space equipped with an action $\pi : U\mathfrak{g} \to \mathrm{End}(M)$ in the category of associative algebras.

One has a one-to-one correspondence between $\mathfrak{g}$-modules and $U\mathfrak{g}$-modules: use $\mathfrak{gl}(M) = \mathrm{Lie}(\mathrm{End}(M))$ and (U, Lie) is an adjoint pair to get

$$\mathrm{Hom}(\mathfrak{g}, \mathfrak{gl}(M)) = \mathrm{Hom}(U(\mathfrak{g}), \mathrm{End}(M)). \tag{3}$$

Remark 3.1. Given any associative algebra A, an A-module M is a morphism $\pi : A \to \mathrm{End}(M)$ in the category of k-associative algebras. Applying Lie then makes M a $\mathrm{Lie}(A)$-module:

$$\mathrm{Lie}(A) \to \mathfrak{gl}(M) \tag{4}$$

3.3 Topological G-modules

A *topological G-module* or simply *G-module* is a $\mathbb{C}-$vector space M (locally convex Hausdorff) equipped with a *continuous* action $\pi : G \to \mathrm{Aut}(M)$. Let $\mathcal{C}_G$ be the category of topological G-modules and equivariant continuous linear maps.

3.4 Smooth G-modules

Let M be a topological G-module. A vector $v \in M$ is *smooth* if it is fixed by an open subgroup of G. The set of smooth vectors in M defines a G-submodule $M^\infty \subset M$. A *smooth G-module* is a topological G-module M satisfying

1. $M^\infty = M$,
2. $v \mapsto (g \mapsto gv) : \ M \to C^\infty(G, M)$ is a homeomorphism onto its image.

The full subcategory of smooth G-modules (in $\mathcal{C}_G$) is denoted $\mathcal{C}_G^\infty$.

The assignment $M \mapsto M^\infty$ is functorial and gives $\mathcal{C}_G \to \mathcal{C}_G^\infty$.

If M is smooth we can differentiate and obtain $(d\pi)_1 : \mathfrak{g} \to \mathfrak{gl}(M)$. This makes M a $\mathfrak{g}$-module. Conversely, if G is *simply connected*, then there is an inverse functor associating to a $\mathfrak{g}$-module M a smooth G-module. Thus, there is a one-to-one correspondence between smooth G-modules and $\mathfrak{g}$-modules.

3.5 Adjoint representations

As an example we discuss the adjoint representation. Every Lie algebra $\mathfrak{g}$ is a module over itself via the *adjoint representation* ad, which is simply $\mathfrak{g}$ acting on $\mathfrak{g}$ by left-multiplication:

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) && (5)\\ x &\mapsto [x, \cdot]. && (6) \end{aligned}$$

Indeed this is a Lie algebra homomorphism (note that the Lie bracket for $\mathfrak{gl}(\mathfrak{g})$ is given by the commutator $[x, y] = xy - yx$):

$$\begin{aligned} [\mathrm{ad}_x, \mathrm{ad}_y] &= \mathrm{ad}_x \mathrm{ad}_y - \mathrm{ad}_y \mathrm{ad}_x && (7)\\ &= [x, [y, \cdot]] - [y, [x, \cdot]] && (8)\\ &= -[[y, \cdot], x] - [[\cdot, x], y] && (9)\\ &= [[x, y], \cdot] && (10)\\ &= \mathrm{ad}_{[x,y]} && (11) \end{aligned}$$

If $k \in \{\mathbb{R}, \mathbb{C}\}$ and G a Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, then this $\mathfrak{g}$-module structure on $\mathfrak{g}$ lifts[6] to a G-module structure. This G-module is also called the *adjoint representation*, denoted by $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ and defined by $\mathrm{Ad}(x) = (dc_x)_1$, where $c_x : y \mapsto xyx^{-1}$ is conjugation by x on G.

3.6 $(\mathfrak{g}, K)$-modules

Let M be a G-module. Then M^∞ also has induced the structure of a $\mathfrak{g}$-module and in the case of simply connected G one can reconstruct the G-action on M^∞ from its induced $\mathfrak{g}$-action. We want to simplify the situation by restricting M to an even smaller subspace, but also remembering the action of a subgroup $K < G$. We say $v \in M$ is *K-finite* if it is contained in a finite dimensional K-submodule. The space of K-finite vectors is denoted $V_{(K)}$. For $K = G$ and $V_{(K)} = M$, we call M *locally finite*. We are mostly interested in *compact* subgroups K. Then $V_{(K)}$ is a *semi-simple* K-module. Moreover, if K is *maximal compact*, then $V_0 := M^\infty \cap V_{(K)}$[7] also comes with an $\mathfrak{g}$-action and both actions (the one of K and the one of $\mathfrak{g}$) are compatible: for $k \in K$, $X \in \mathfrak{g}$ and $v \in V_0$ we have

$$k(Xv) = (kXk^{-1})kv = \mathrm{Ad}(k)(X)kv. \qquad (12)$$

[6]By this we mean $\mathfrak{g} = T_1(G)$ and $\mathrm{ad} = (d\mathrm{Ad})_1$.

[7]If M is smooth this extra step of filtering the smooth vectors is unnecessary.

This module V_0 is an example of what is called a $(\mathfrak{g}, K)$-module: A $(\mathfrak{g}, K)$-*module* is a real or complex vector space M that is both: a $\mathfrak{g}$-module and a locally finite and semi-simple K-module, satisfying the compatibility conditions:

1. $kXv = (kXk^{-1})kv = \mathrm{Ad}(k)(X)kv$ for all $k \in K$, $X \in U\mathfrak{g}$ and $v \in M$,
2. if $U \subset M$ is a finite dimensional K-submodule, then the representation π of K on U is differentiable and its differential $(d\pi)_1$ coincides with the action π of $\mathfrak{g}$ restricted to $\mathfrak{k} = T_1 K$:

$$(d\pi)_1 = \pi_{|\mathfrak{k}}. \tag{13}$$

The category of $(\mathfrak{g}, K)$-modules and $(\mathfrak{g}, K)$-morphisms is denoted $\mathcal{C}_{\mathfrak{g},K}$.

Remark 3.2. One can think of V_0 equipped with the actions of K and $\mathfrak{g}$ as an infinitesimal approximation to the G-module M. Indeed one defines smooth G-modules M, M' to be *infinitesimally equivalent* if their associated $(\mathfrak{g}, K)$-modules are isomorphic.

4 Cohomology

The following motivation is from [1]:

> If G is a *compact simply connected* Lie group, then it is determined by its Lie algebra, so it should be possible to calculate its cohomology from the Lie algebra. This can be done as follows. Its cohomology is the de Rham cohomology of the complex of differential forms on G. Using an averaging process, this complex can be replaced by the complex of left-invariant differential forms. The left-invariant forms, meanwhile, are determined by their values at the identity, so that the space of left-invariant differential forms can be identified with the exterior algebra of the Lie algebra, with a suitable differential.
>
> The construction of this differential on an exterior algebra makes sense for any Lie algebra, so is used to define Lie algebra cohomology for all Lie algebras. More generally one uses a similar construction to define Lie algebra cohomology with coefficients in a module.
>
> It should be noted that if G is a simply connected *noncompact* Lie group, the Lie algebra cohomology of the associated Lie algebra $\mathfrak{g}$ does not necessarily reproduce the de Rham cohomology of G. The reason for this is that the passage from the complex of all differential forms to the complex of left-invariant differential forms uses an averaging process that only makes sense for compact groups.

4.1 (Co)homology of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over a field k and M be a left $\mathfrak{g}$-module. As the usual story goes we try to derive the following functors:

- the *invariant* submodule functor $M \mapsto M^{\mathfrak{g}} = \{m \in M \mid \forall X \in \mathfrak{g} : \ Xm = 0\}$,
- The *coinvariant* quotient module functor $M \mapsto M_{\mathfrak{g}} = M/\mathfrak{g}M$.

We then define *homology* and *cohomology* groups of $\mathfrak{g}$ with coefficients in M by

$$\mathrm{H}_n(\mathfrak{g}, M) = L_n(\cdot_{\mathfrak{g}})(M) \tag{14}$$
$$\mathrm{H}^n(\mathfrak{g}, M) = R^n(\cdot^{\mathfrak{g}})(M). \tag{15}$$

By definition:

$$\mathrm{H}_0(\mathfrak{g}, M) = M_{\mathfrak{g}} \tag{16}$$
$$\mathrm{H}^0(\mathfrak{g}, M) = M^{\mathfrak{g}}. \tag{17}$$

Actually one has to make sure that there are enough injectives and enough projectives for those definitions to work. The case with projectives is clear: take free resolutions.

Recall that $U\mathfrak{g}$ is the quotient of $T\mathfrak{g}$ by the ideal generated by elements of the form $i([x,y]) - i(x)i(y) + i(y)i(x)$ with $i : \mathfrak{g} \to T\mathfrak{g}$. The k-algebra homomorphism $\epsilon : U\mathfrak{g} \to k$ sending $i(\mathfrak{g})$ to zero is called *augmentation* and its kernel $\mathfrak{J}$ *augmentation ideal*

$$0 \to \mathfrak{J} \to U\mathfrak{g} \to k \to 0. \tag{18}$$

Proposition 4.1. Let k be equipped with the trivial $\mathfrak{g}$-module structure. We can compute Lie algebra cohomology as $U\mathfrak{g}$-cohomology:

$$H_n(\mathfrak{g}, M) = Tor_n^{U\mathfrak{g}}(k, M) \tag{19}$$
$$H^n(\mathfrak{g}, M) = Ext^n_{U\mathfrak{g}}(k, M). \tag{20}$$

Proof. Derived functors are isomorphic if their underlying functors are:

$$M_{\mathfrak{g}} = M/\mathfrak{g}M = M/\mathfrak{J}M = U\mathfrak{g}/\mathfrak{J} \otimes_{U\mathfrak{g}} M = k \otimes_{U\mathfrak{g}} M \tag{21}$$
$$M^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(k, M) = \mathrm{Hom}_{U\mathfrak{g}}(k, M). \tag{22}$$

□

Proposition 4.2. Let M, N be left $\mathfrak{g}$-modules. Then $\mathrm{Hom}_k(M, N)$ can be made a $\mathfrak{g}$-module via $Xf(m) = Xf(m) - f(Xm)$ for $X \in \mathfrak{g}$, $m \in M$. With this one has:

$$\mathrm{Ext}^n_{U\mathfrak{g}}(M, N) = \mathrm{H}^n(\mathfrak{g}, \mathrm{Hom}_k(M, N)). \tag{23}$$

Proof. This follows from $\mathrm{Hom}_{\mathfrak{g}}(M, N) = \mathrm{Hom}_k(M, N)^{\mathfrak{g}}$ and the fact that for a field k, there are no nontrivial extensions of k-modules. □

4.2 Homology of Lie algebras in degree one

To understand the homology in low degrees we look at the long exact sequence for

$$0 \to \mathfrak{J} \to U\mathfrak{g} \to k \to 0 \tag{24}$$

and $Tor_n^{U\mathfrak{g}}(\cdot, M)$ (M a $\mathfrak{g}$-module):

$$\cdots \to Tor_1^{U\mathfrak{g}}(\mathfrak{J}, M) \to Tor_1^{U\mathfrak{g}}(U\mathfrak{g}, M) \to Tor_1^{U\mathfrak{g}}(k, M) \tag{25}$$

$$\to \mathfrak{J} \otimes_{U\mathfrak{g}} M \to U\mathfrak{g} \otimes_{U\mathfrak{g}} M \to k \otimes_{U\mathfrak{g}} M \to 0. \tag{26}$$

Since $U\mathfrak{g}$ is free as $U\mathfrak{g}$-module one has $Tor_n^{U\mathfrak{g}}(U\mathfrak{g}, M) = 0$ for $n \geq 1$. Hence, for $n \geq 2$

$$\mathrm{H}_n(\mathfrak{g}, M) = Tor_n^{U\mathfrak{g}}(k, M) = Tor_{n-1}^{U\mathfrak{g}}(\mathfrak{J}, M) \tag{27}$$

and

$$0 \to \mathrm{H}_1(\mathfrak{g}, M) \to \mathfrak{J} \otimes_{U\mathfrak{g}} M \to M \to M_{\mathfrak{g}} \to 0. \tag{28}$$

Finally, tensoring 24 with $\mathfrak{J} \otimes_{U\mathfrak{g}}$ and using that $i : \mathfrak{g} \to U\mathfrak{g}$ maps $[\mathfrak{g}, \mathfrak{g}]$ to $\mathfrak{J}^2$, inducing $\mathfrak{g}^{\mathrm{ab}} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = \mathfrak{J}/\mathfrak{J}^2$ yields

$$\mathfrak{J} \otimes_{U\mathfrak{g}} k = \mathfrak{g}^{\mathrm{ab}}. \tag{29}$$

Theorem 4.3. *If M is a trivial $\mathfrak{g}$-module, then $\mathrm{H}_1(\mathfrak{g}, M) = \mathfrak{g}^{\mathrm{ab}} \otimes_k M$.*

Proof. This follows from 28 and $M_{\mathfrak{g}} = M$:

$$\mathrm{H}_1(\mathfrak{g}, M) = \mathfrak{J} \otimes_{U\mathfrak{g}} M = (\mathfrak{J} \otimes_{U\mathfrak{g}} k) \otimes_k M = \mathfrak{J}/\mathfrak{J}^2 \otimes_k M = \mathfrak{g}^{\mathrm{ab}} \otimes_k M. \tag{30}$$

□

4.3 Cohomology of Lie algebras in degree one

Theorem 4.4. *The first cohomology of a $\mathfrak{g}$-module M is the quotient of derivations $Der(\mathfrak{g}, M)$ modulo inner derivation $Der_{in}(\mathfrak{g}, M)$:*

$$\mathrm{H}^1(\mathfrak{g}, M) = Der(\mathfrak{g}, M)/Der_{in}(\mathfrak{g}, M). \tag{31}$$

A proof is easy after we introduce the Chevalley-Eilenberg complex, later.

Recall, that a *derivation* D is a k-linear map $D : \mathfrak{g} \to M$, satisfying the *Leibniz law* (where $\mathfrak{g}$ is considered a $\mathfrak{g}$-module via its adjoint representation: "$x \cdot y = [x, y]$"): $D([x, y]) = xDy - yDx$. $Der(\mathfrak{g}, M)$ is the k-submodule of $\mathrm{Hom}_k(\mathfrak{g}, M)$ of derivations. The subspace $Der_{in}(\mathfrak{g}, M)$ of *inner derivations* is defined as the image of

$$M \to Der(\mathfrak{g}, M) \tag{32}$$

$$m \mapsto D_m : X \mapsto Xm. \tag{33}$$

4.4 The Chevalley-Eilenberg complex

In this subsection we want to construct a complex $C_*(\mathfrak{g})$ satisfying:

Theorem 4.5. *The* Chevalley-Eilenberg complex $C_*(\mathfrak{g}) = U\mathfrak{g} \otimes_k \wedge^p \mathfrak{g}$ *is a projective resolution of the* $\mathfrak{g}$*-module* k*. In particular, if* M *is a* $\mathfrak{g}$*-module we can compute its (co)homology via:*

$$\mathrm{H}_n(\mathfrak{g}, M) = H_n(M \otimes_{U\mathfrak{g}} C_*(\mathfrak{g})), \tag{34}$$

$$\mathrm{H}^n(\mathfrak{g}, M) = H^n(\mathrm{Hom}_{\mathfrak{g}}(C_*(\mathfrak{g}), M)). \tag{35}$$

Proof. Construction: First note that every $C_n(\mathfrak{g})$ is actually a *free* $U\mathfrak{g}$-module. We define the differentials as follows: For low degrees we use

$$C_1(\mathfrak{g}) = U\mathfrak{g} \otimes_k \mathfrak{g} \xrightarrow{d} C_0(\mathfrak{g}) = U\mathfrak{g} \otimes_k k = U\mathfrak{g} \xrightarrow{\epsilon} k \to 0 \tag{36}$$

with $d(u \otimes X) = uX$ and ϵ being augmentation. Note that $im(d) = \mathfrak{J}$, hence so far the sequence is exact. For higher degrees $n \geq 2$ we define

$$d : C_n(\mathfrak{g}) \to C_{n-1}(\mathfrak{g}) \tag{37}$$

$$u \otimes X_1 \wedge \cdots \wedge X_n \mapsto \theta_1 + \theta_2, \tag{38}$$

where

$$\theta_1 = \sum_{i=1}^{n} (-1)^{i+1} uX_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n \tag{39}$$

$$\theta_2 = \sum_{i<j} (-1)^{i+j} u \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n. \tag{40}$$

For example, if $n = 2$, then $d(u \otimes X \wedge Y) = uX \otimes Y - uY \otimes X - u \otimes [X, Y]$. □

Corollary 4.6. If $\mathfrak{g}$ is n-dimensional over k, then (co)homology in degree $> n$ must vanish.

Proof. Then, $\wedge^n \mathfrak{g} = 0$. □

Remark 4.7. We have

$$M \otimes_{U\mathfrak{g}} C_*(\mathfrak{g}) = M \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes_k \wedge^* \mathfrak{g} = M \otimes_k \wedge^* \mathfrak{g} \tag{41}$$

$$\mathrm{Hom}_{\mathfrak{g}}(C_*(\mathfrak{g}), M) = \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes_k \wedge^* \mathfrak{g}, M) = \mathrm{Hom}_k(\wedge^* \mathfrak{g}, M) \tag{42}$$

and in the latter an n-cochain is simply an *alternating* k*-multilinear function* $f(X_1, \ldots, X_n)$ on $\mathfrak{g}^n$ with values in M. Its coboundary is

$$df(X_1, \ldots, X_{n+1}) = \sum (-1)^i X_i f(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1}) \tag{43}$$

$$+ \sum (-1)^{i+j} f([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{n+1}). \tag{44}$$

Example 4.8. Check that

$$Z^1(\mathfrak{g}, M) = Der(\mathfrak{g}, M) \quad (45)$$

$$B^1(\mathfrak{g}, M) = Der_{in}(\mathfrak{g}, M) \quad (46)$$

and hence $\mathrm{H}^1(\mathfrak{g}, M) = Der(\mathfrak{g}, M)/Der_{in}(\mathfrak{g}, M)$ as claimed.

4.5 Cohomology of Lie algebras in degrees two and three

Theorem 4.9. *Let M be a $\mathfrak{g}$-module. Then $\mathrm{H}^2(\mathfrak{g}, M)$ is in bijection with the set $Ext(\mathfrak{g}, M)$ of equivalence classes of extensions of $\mathfrak{g}$ by M[8]. Similar, $\mathrm{H}^3(\mathfrak{g}, M)$ is in bijection with 2-extensions, ie. four term short exact sequences of Lie algebras*

$$0 \to M \to \mathfrak{h} \to \mathfrak{f} \to \mathfrak{g} \to 0. \quad (47)$$

Proof. I think this is a standard argument from homological algebra. The keyword is: *Yoneda extension.* □

5 Relative cohomology

5.1 Cohomology of $(\mathfrak{g}, \mathfrak{k})$-modules

As a motivation for relative cohomology there is a result of E. Cartan, stating that for G compact connected, one has

$$\mathrm{H}^*(\mathfrak{g}, \mathfrak{k}; M) = \mathrm{H}^*(G/K, M), \quad (48)$$

where the left side denotes relative cohomology of $\mathfrak{g}$ modulo $\mathfrak{k}$, where $\mathfrak{k} = \mathrm{Lie}(K)$.

As usual it would be nice to have two characterizations: one abstract as a cohomology of a derived functor, the other concrete in terms of cohomology of a complex. Indeed, in terms of $(U\mathfrak{g}, U\mathfrak{k})$-*algebras* one has

$$\mathrm{Ext}^q_{U\mathfrak{g}, U\mathfrak{k}}(k, M) = \mathrm{H}^q(\mathfrak{g}, \mathfrak{k}; M) \quad (49)$$

and for the "cochain complex version" we define a subcomplex of the Chevalley-Eilenberg complex by

$$C^q(\mathfrak{g}, \mathfrak{k}; M) = \mathrm{Hom}_{\mathfrak{k}}(\wedge^q(\mathfrak{g}/\mathfrak{k}), M) \quad (50)$$

with the $\mathfrak{k}$-action on $\wedge^q(\mathfrak{g}/\mathfrak{k})$ being induced from its adjoint action on $\mathfrak{k}$. In other words, it is the subspace of $\mathrm{Hom}_k(\wedge^q(\mathfrak{g}/\mathfrak{k}), M)$ of elements f satisfying (for $X \in \mathfrak{k}$, $X_i \in \mathfrak{g}/\mathfrak{k}$, $i = 1, \dots, q$):

$$\sum_i f(X_1, \dots, [X, X_i], \dots, X_q) = Xf(X_1, \dots, X_q). \quad (51)$$

The cohomology groups $\mathrm{H}^*(\mathfrak{g}, \mathfrak{k}; M)$ of $C^*(\mathfrak{g}, \mathfrak{k}; M)$ are called *relative cohomology groups* of $\mathfrak{g}$ mod $\mathfrak{k}$ with coefficients in M.

[8] Actually, $Ext(\mathfrak{g}, M)$ also admits an addition given by the Baer sum and the bijection stated then becomes an isomorphism.

5.2 Cohomology of $(\mathfrak{g}, K)$-modules

Another relative cohomology considers the case with $(\mathfrak{g}, K)$-modules M. For this let K be a *maximal compact* subgroup of the real Lie group G[9] and let $k = \mathbb{R}$.

We only give a cochain computing cohomology: Let

$$C^q(\mathfrak{g}, K; M) = \mathrm{Hom}_K(\wedge^q(\mathfrak{g}/\mathfrak{k}), M), \tag{52}$$

where this time K acts on $\mathfrak{g}/\mathfrak{k}$ via the adjoint representation. Note the relation to $(\mathfrak{g}, \mathfrak{k})$-module cohomology ($K^0$ is the identity component of K)

$$C^q(\mathfrak{g}, K; M) \subset C^q(\mathfrak{g}, K^0; M) = C^q(\mathfrak{g}, \mathfrak{k}; M). \tag{53}$$

Also K/K^0 naturally acts on $C^q(\mathfrak{g}, \mathfrak{k}; M)$ and we have

$$C^q(\mathfrak{g}, K; M) = C^q(\mathfrak{g}, \mathfrak{k}; M)^{K/K^0}. \tag{54}$$

$C^*(\mathfrak{g}, K; M)$ is a subcomplex of $C^*(\mathfrak{g}, \mathfrak{k}; M)$ and it follows from 54 that:

$$\mathrm{H}^q(\mathfrak{g}, K; M) = \mathrm{H}^q(\mathfrak{g}, \mathfrak{k}; M)^{K/K^0} \tag{55}$$

6 Application: Weyl's theorem

Goal of this section is

Theorem 6.1 (Weyl's theorem)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field k of characteristic 0. Then every finite dimensional $\mathfrak{g}$-module is semisimple, ie. a direct sum of simple $\mathfrak{g}$-modules.*

Proof. Sketch!:

Step 1: first proof a vanishing theorem: If M is *simple* and $\neq k$, then for all i

$$\mathrm{H}^i(\mathfrak{g}, M) = \mathrm{H}_i(\mathfrak{g}, M) = 0. \tag{56}$$

The proof for this uses *Schur's lemma* and the so called *Casimir operator*.

Step 2: Assume the opposite, ie. that M is not semisimple, ie. a direct sum of simple modules. Then, since M is finite dimensional, there is a submodule M_1 minimal with the property "not being semisimple". Clearly, M_1 is not simple, so it contains a proper submodule M_0. By minimality, M_0 and $M_2 = M_1/M_0$ *are* semisimple, while M_1 is not. This means, that

$$0 \to M_0 \to M_1 \to M_2 \to 0 \tag{57}$$

is not split and hence induces a nontrivial element in

$$\mathrm{Ext}^1_{U\mathfrak{g}}(M_2, M_0) = \mathrm{H}^1(\mathfrak{g}, \mathrm{Hom}_k(M_2, M_0)). \tag{58}$$

[9] G is assumed to have a finite component group.

Step 3: We are left with showing that such an element does not exist. This follows from *Whitehead's first lemma*[10]: let $\mathfrak{g}$, k be as in our setting and M be a finite dimensional $\mathfrak{g}$-module. Then

$$\mathrm{H}^1(\mathfrak{g}, M) = 0. \tag{59}$$

Use $\mathrm{Hom}_k(M_2, M_0)$ for M to reach the contradiction. □

References

[1] Wikipedia authors, `https://en.wikipedia.org/wiki/Lie_algebra_cohomology`

[2] Shen-Ning Tung, *Lie Algebra Homology and Cohomology*, 2013

[3] A. Borel, N. Wallach, *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, Princeton, N.J.: Princeton University Press, 1980.

[10] Here is a proof: If M is simple, then either $M = k$ and $\mathrm{H}^1(\mathfrak{g}, k) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 0$ or else $M \neq k$ and $\mathrm{H}^*(\mathfrak{g}, M) = 0$ by the vanishing theorem. Otherwise, M contains a proper submodule L and by induction $\mathrm{H}^1(\mathfrak{g}, L) = \mathrm{H}^1(\mathfrak{g}, M/L) = 0$. Since $0 \to L \to M \to M/L \to 0$ induces an exact sequence of cohomology: $\cdots \to \mathrm{H}^1(\mathfrak{g}, L) \to \mathrm{H}^1(\mathfrak{g}, M) \to \mathrm{H}^1(\mathfrak{g}, L/M) \to \dots$, we have $\mathrm{H}^1(\mathfrak{g}, M) = 0$.