Darmstadt - Frankfurt seminar

The paramodular conjecture

Wintersemester 2019

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Langlands program predicts deep connections between geometric and automorphic objects encoded in associated L-functions and Galois representations. One of its instances is the famous modularity theorem, which provides a relation between elliptic curves of conductor N and weight 2 level N cuspidal Hecke eigenforms via equality of the L-functions; in particular, the associated Galois representations are equivalent. A similar statement for abelian surfaces is known as the paramodular conjecture.

Abelian surfaces may be divided into the following categories, depending on how they arise or the nature of their endomorphism ring:

- products of two elliptic curves,
- of GL_2-type,
- potentially of GL_2-type, i.e. Weil restrictions of elliptic curves,
- "typical" surfaces, having minimal endomorphisms defined over $\bar{\mathbb{Q}}$.

As we will see, in the first three cases the paramodular conjecture holds and the corresponding automorphic objects arise as various types of Yoshida lifts. The last case is still open, but is now confirmed by a few genuine examples; we will look closer at one of them. **The goal** of the seminar is to get the idea of the methods and the tools which were used to obtain these results.

The goal of first two talks is to give an idea of the place of the paramodular conjecture within Langlands framework and indicate the notions that will be discussed later in more detail. Depending on the background, they are not necessarily difficult. The talks 2.1, 3.1, 3.2 are very much introductory and can be presented by PhD or master students, although for 3.1 and 3.2 it may help to have background in algebraic geometry. Similarly, the talks 5.1-5.2 (on Faltings-Serre method) and 6.1 (computational aspects of Siegel modular forms) are almost not related to the previous talks and can be presented by PhD students; although some knowledge on modular forms would be helpful for the last one. Probably the hardest are the talks 4.1 and 4.2. The first one uses the result of Khare and Wintenberger on Serre's modularity conjecture together with work of Ribet (very nicely written!) to prove modularity of abelian surfaces of GL_2-type; and the latter associates Galois representations to Siegel modular forms of degree 2.

1 Darmstadt 7.11.2019, speakers: Brandon, Jolanta

1.1 Big picture

Expand section 1 from the notes [15] to formulate the modularity conjecture for a general abelian variety. Present specialization to the case of elliptic curves. Explain briefly what elliptic curve is, what is the associated L-function and what is the modular form occurring there (see for example [8]).

1.2 Paramodularity

Survey sections 2.1-2.2 from the notes [15]: this is a specialization of the previous talk to abelian surfaces. In particular, formulate "The Precise Paramodular Conjecture" and define/characterize paramodular local newforms, tempered and generic representations (for this consult [4] or [9]). For more details see [13]; see especially p. 2415 there which explains the appearance of weight 2 in the paramodular conjecture.

2 Frankfurt 21.11.2019, speakers: Riccardo, Priyanka

2.1 Paramodular forms

Define paramodular forms and explain how to associate with them an automorphic representation (see [12, p. 3-5] or [9, beginning of section 6]). Define newforms and oldforms as in [12, section 5] and explain the link with the paramodular newforms defined in the previous talk. Deduce the classical paramodular conjecture from the "The Precise Paramodular Conjecture" following [15, section 2.3]. Define Hecke operators $T(p)$ and $T_1(p^2)$, and spinor Euler factors as in [3, section 4.2]. (The conjectures from [12, section 5] are proven in [14].)

2.2 Yoshida lifting

Review the properties of the Yoshida lifting as described in [15, section 3] and use Theorem 3.1 and Proposition 3.2 there to prove paramodularity for a product of two elliptic curves following reasoning in [15, beginning of section 4]. Many notions mentioned in [15, section 3] are defined in [4].

3 Darmstadt 28.11.2019, speakers: Jaro, Theresa

3.1 Abelian varieties

Define the following notions (using e.g. the lecture notes [7]: introduction and sections I.1, I.7): abelian variety, isogeny and its degree (with n_A from [7, Theorem 7.2] as an example), Tate module, polarization (if possible). Define also Jacobian of a smooth projective curve [7, section III.1], state Theorem 10.1 from [7] (without a proof), Theorem 3.4.11 from [2] (without a proof; especially in case $m = 1, \mathfrak{o} = \mathbb{Z}$; see also p. 4 there and compare with our leading example from [3, Theorem 1.2.1]). Furthermore, if time permits, look at [3, the beginning of section 4.1] in order to define a "typical" variety and prove Lemma 4.1.1 and 4.1.2.

3.2 ... and the associated Galois representations

Following [3, section 4.1]: explain how to associate a Galois representation associated to an abelian surface (it may be helpful to look first and survey some of the background information from [6, sections 7.1-7.2]), define the local L-factors and state its connection with the zeta function (as defined for example in [7, p. 78]). Define also places of good and potential reduction of the variety and its conductor: see [17, sections 1-3].

Note: Keep in mind the specific example of an abelian variety we want to consider: the one from [3, Theorem 1.2.1]. The speakers should feel free to distribute the topics listed above differently.

4 Frankfurt 16.01.2020, speakers: Michalis, Jolanta

4.1 Special cases of paramodularity

Explain the paramodularity conjecture for abelian surfaces of GL_2-type and Weil restrictions of elliptic curves. Start with an overview in [15, section 4], then look at [5] and [11]. It may be also helpful to look at [6, section 7.3].

4.2 Galois representations associated to Siegel modular forms

Survey [3, section 4.3]: introduce the missing notions so that you can state/explain all the results there. For a (big) simplification of [3, Theorem 4.3.4] and the story around it, see [6, sections 9.1-9.2]. More detailed description of the construction of the Galois representations together with an extensive list of references can be found in [18, sections 4.1, 4.2 and 4.4].

5 Darmstadt 30.01.2020, speakers: Patrick B., Patrick H.

5.1 Faltings-Serre method

Describe application of Faltings-Serre method in [3]. Read first about the original application to elliptic curves in [16] to get the feeling for the method (in French, but only 2 pages). Then move to [3, sections 2.1-2.3]. You can refer in your talk to the case of elliptic curves if this makes the method easier to understand.
Goal: Algorithm 2.2.3 (with the proof of correctness), Algorithm 2.4.1 (with an idea for the proof of correctness)

5.2 ... and the computational aspects behind it

Survey the sections 3.1-3.3 and 5.1-5.3 in [3] in order to be able to state Theorem 5.3.3. The sections 3.1-3.3 develop the theory needed to carry out the steps 2-4 from the Algorithm 2.4.1, and the sections 5.1-5.3 comprise practical application of it in case of the steps 2-3. In case of lack of time you can skip section 3.3.

6 Frankfurt 13.02.2020, speakers: Thomas, Yingkun

6.1 Computing Hecke eigenvalues of Siegel modular forms

The main source for this talk is [3, section 6]. Define briefly Gritsenko lifts (mention also their appearance in the paramodular conjecture, as stated for example in [2]) and theta blocks so that you can tell the meaning of the equation (6.2.2) in [3]. Then explain how the authors compute the Hecke eigenvalues of paramodular newforms: find a balance between describing a general method and focusing on the example of f_{277}.

6.2 Paramodularity at level $N = 277$ and beyond

Prove Theorem 1.2.1 in [3] following the steps described in [3, section 7.1]. If the time permits comment (following [2, section 8]) why the conjecture as stated in [3] does not give a bijection between suitable paramodular forms and isogeny classes of abelian surfaces. Finally, comment on (potential) modularity results obtained in [1], especially Theorems 1.1.5 and 1.1.7 (see the introduction there).

References

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