Uniformity of rational points on curves

DaFra-Seminar – Sommersemester 2019

Organizers: Annette Werner and Martin Ulirsch

Let X be a smooth projective curve defined over $\mathbb{Q}$. When the genus g of X is ≥ 2, Faltings' Theorem tells us that the number $\#X(\mathbb{Q})$ of $\mathbb{Q}$-rational points on X is finite. The following conjecture is refinement of Falting's Theorem.

Conjecture (Uniformity). *For every $g \geq 2$, there is a constant $B(g, \mathbb{Q})$ such that for every smooth and projective curve over $\mathbb{Q}$ of genus g we have:*

$$\#X(\mathbb{Q}) \leq B(g, \mathbb{Q}) \ .$$

The uniformity conjecture would follow from the currently still open Lang-Vojta-conjecture, which is a higher-dimensional analogue of Faltings' Theorem. It was considered somewhat outrageous for a while and was taken as an indication that the Lang-Vojta conjecture was false.

In [KRZB16], however, Katz–Rabinoff–Zureick-Brown succesfully combined techniques from tropical and non-Archimedean geometry with the method of Coleman-Chabautry in order to prove the following Theorem, which proves the conjecture for curves whose Jacobian has rank $\leq g - 3$.

Theorem (Katz–Rabinoff–Zureick-Brown, [KRZB16]). *Let X be any smooth curve of genus g defined over $\mathbb{Q}$ and write $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. If $r \leq g - 3$, then:*

$$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28 \ .$$

The goal of this seminar is to understand the proof of this theorem with a particular focus on the methods from tropical and non-Archimedean geometry that allow us to extend the method of Chabauty-Coleman to curves with bad reduction. After covering the background, we will mostly follow the survey [KRZB18] written by the authors.

List of talks

Talks 1-6 are particularly suitable for Master- and Ph.D.-students, since they require less background. Talk 5 and 6 do not depend on Talk 1-4. One of the two speakers for Talk 7-8 should know about non-Archimedean uniformization of abelian varieties. Talk 9 is independent from the talks before and might be a good choice for a young number theorist/arithmetic geometer. Talk 10-12 require the most technical insight and might be a good choice for more experienced speakers.

The talks naturally come in the following subgroups: Talk 1-4, Talk 5-6, Talk 7-8 (which depend on Talk 1-5), and Talk 10-12 (which depend on everything before). The speakers within these groups should communicate with each other to make sure they know what has already been covered and what still needs to be done.

Talk 1: Berkovich analytic spaces (25.4.)

Introduce Berkovich analytic spaces focussing on the topological structure of analytifications, as e.g. in [Wer13], [Bak08], and [Pay15]. The central points are the following: Recall non-Archimedean non-Archimedean absolute values as in [Wer13, Section 2], then describe the general theory, as in [Wer13, Section 5]. Conclude with [Wer13, Section 3 and 4] as an extended example. The central goal is to understand the picture of the Berkovich analytic line, see [Bak08, Figure 1].

Talk 2: Skeletons of curves I (25.4)

Recall the notion of an affinoid algebra from Talk 1 and explain the construction of skeletons of generalized annuli in [BPR13, 2.1-2.7]. Then explain the notion of a semistable vertex set and of the skeleton associated to such a semistable vertex set, as in [BPR13, 3.1-3.9].

Talk 3: Skeletons of curves II (9.5.)

Recall the construction of the skeleton associated to a semistable vertex set from [BPR13, Section 3] and then survey [BPR13, Section 4] that explains how semistable vertex sets are related to semistable models. The central point is the reduction map and Theorem 4.11, which should be explained in detail. Please state the Stable Reduction Theorem [BPR13, Theorem 4.22] and explain its interpretation in the language developed before.

Talk 4: Metric structures and the slope formula (9.5.)

"Explain" the metric structure on the skeleton. The relevant parts in are: Section [BPR13, 2.8–end], [BPR13, Definition 3.10–end] and [BPR13, Section 5]. Finish with the slope formula [BPR13, Theorem 5.15] that will be used in Talk 7/8. This can be complemented with the original (and beautifully written) paper [BPR16] and, in particular, [BPR16, Theorem 5.14.].

Talk 5: Divisor theory on metric graphs (23.5.)

Survey the divisor theory on metric graphs, following e.g. [BJ16, Part 1]. You will have to pick and choose so that you can fit everything into your talk. The central notions to be covered are: Divisors on finite and metric graphs, rational equivalence, Jacobians, and Abel-Jacobi maps (as in [BJ16, Section 2.2, 3.2, and 3.3]). Please state the Abel-Jacobi-Theorem, which will be proved in Talk 6.

Talk 6: Jacobians of metric graphs (23.5)

Survey the proof of the Abel-Jacobi-Theorem [BF11, Theorem 3.4] for metric graphs. The original account can be found in [MZ08]. The central ingredients of the proof in [BF11] are Theorem 2.8, Theorem 2.11, and its refinement, Corollary 3.11. Section 4-6 are interesting, but can safely be ignored.

Talk 7/8: Skeletons of Jacobians I/II (13.6.)

State and prove the main result of [BR15], Theorem 1.3. It states that the non-Archimedean skeleton of the Jacobian of a curve X is equal to the Jacobian of the skeleton of X and that the retraction to the skeleton naturally commutes with Abel-Jacobi maps. First focus on [BR15, Section 2], where an overview over the proof is given. Then carefully explain the blackbox "non-Archimedean uniformization of abelian varieties", as in [BR15, Section 4]. Finally, survey [BR15, Section 5 and 6] to wrap up the proof. In particular, explain how the slope formula from Talk 4 is used in the proof of [BR15, Proposition 5.3].

Talk 9: The method of Chabauty-Coleman I: An overview (27.6.)

Give an overview of the method of Chabauty-Coleman following [MP12]. This talk is independent from all talks before and is an ideal choice for a young number theorist/arithmetic geometer. The central parts of [MP12] are Section 4 and 5 as well as a selection of the examples in Section 8.

Talk 10: The method of Chabauty-Coleman II: Theories of p-adic integration (27.6.)

Survey [KRZB18, Section 4] which explains how the previous tropical and non-Archimedean story helps us refine the Chabauty-Coleman version of p-adic integration.

Talk 11/12: Uniformity of rational points on curves I and II (11.7.)

Survey the main results and proofs in [KRZB16], as in [KRZB18, Section 1 and 5].

References

[Bak08] Matthew Baker, *An introduction to Berkovich analytic spaces and non-Archimedean potential theory on curves*, p-adic geometry, Univ. Lecture Ser., vol. 45, Amer. Math. Soc., Providence, RI, 2008, pp. 123–174. MR 2482347

[BF11] Matthew Baker and Xander Faber, *Metric properties of the tropical Abel-Jacobi map*, J. Algebraic Combin. **33** (2011), no. 3, 349–381. MR 2772537

[BJ16] Matthew Baker and David Jensen, *Degeneration of linear series from the tropical point of view and applications*, Nonarchimedean and tropical geometry, Simons Symp., Springer, [Cham], 2016, pp. 365–433. MR 3702316

[BPR13] Matthew Baker, Sam Payne, and Joseph Rabinoff, *On the structure of non-Archimedean analytic curves*, Tropical and non-Archimedean geometry, Contemp. Math., vol. 605, Amer. Math. Soc., Providence, RI, 2013, pp. 93–121. MR 3204269

[BPR16] ______, *Nonarchimedean geometry, tropicalization, and metrics on curves*, Algebr. Geom. **3** (2016), no. 1, 63–105. MR 3455421

[BR15] Matthew Baker and Joseph Rabinoff, *The skeleton of the Jacobian, the Jacobian of the skeleton, and lifting meromorphic functions from tropical to algebraic curves*, Int. Math. Res. Not. IMRN (2015), no. 16, 7436–7472. MR 3428970

[Gub13] Walter Gubler, *A guide to tropicalizations*, Algebraic and combinatorial aspects of tropical geometry, Contemp. Math., vol. 589, Amer. Math. Soc., Providence, RI, 2013, pp. 125–189. MR 3088913

[KRZB16] Eric Katz, Joseph Rabinoff, and David Zureick-Brown, *Uniform bounds for the number of rational points on curves of small Mordell-Weil rank*, Duke Math. J. **165** (2016), no. 16, 3189–3240. MR 3566201

[KRZB18] ______, *Diophantine and tropical geometry, and uniformity of rational points on curves*, Algebraic geometry: Salt Lake City 2015, Proc. Sympos. Pure Math., vol. 97, Amer. Math. Soc., Providence, RI, 2018, pp. 231–279. MR 3821174

[MP12] William McCallum and Bjorn Poonen, *The method of Chabauty and Coleman*, Explicit methods in number theory, Panor. Synthèses, vol. 36, Soc. Math. France, Paris, 2012, pp. 99–117. MR 3098132

[MZ08] Grigory Mikhalkin and Ilia Zharkov, *Tropical curves, their Jacobians and theta functions*, Curves and abelian varieties, Contemp. Math., vol. 465, Amer. Math. Soc., Providence, RI, 2008, pp. 203–230. MR 2457739

[Pay15] Sam Payne, *Topology of nonarchimedean analytic spaces and relations to complex algebraic geometry*, Bull. Amer. Math. Soc. (N.S.) **52** (2015), no. 2, 223–247. MR 3312632

[Wer13] Annette Werner, *Non-Archimedean analytic spaces*, Jahresber. Dtsch. Math.-Ver. **115** (2013), no. 1, 3–20. MR 3040415