DAFRA-SEMINAR ON CONDENSED MATHEMATICS

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This is a Zoom learning seminar aiming at giving an introduction to the brand new theory of condensed mathematics after Clausen and Scholze.

Introduction. Condensed mathematics opens a way to develop a theory of "analytic spaces" which unifies different geometries including real and complex manifolds, algebraic geometry (varieties/schemes) and p-adic geometries (rigid/Berkovich/adic spaces). In the seminar we want to understand the foundations of this theory. Its actual construction lies beyond the scope of this seminar and is left for another occasion.

To elaborate, let us consider the field of rational numbers $\mathbb{Q}$. Up to equivalence the possible norms on $\mathbb{Q}$ are given by the (ordinary) absolute value $|\cdot|_\infty$ and for each prime number $p \in \mathbb{N}$ the p-adic value $|\cdot|_p$. Completing with respect to these norms leads to the field of real numbers $\mathbb{R}$ for $|\cdot|_\infty$ and to the field of p-adic numbers $\mathbb{Q}_p$ for $|\cdot|_p$. The resulting geometries are severely different: over $\mathbb{R}$ one obtains Archimedean geometries including Euclidean or hyperbolic geometry, whereas over $\mathbb{Q}_p$ the geometry tends to be very much disconnected, for example, two p-adic discs are either disjoint or concentric. This indicates that the search for a unifying theory is not easy or even unlikely to exist at all, see [Sch2, Lecture 1]. Nevertheless, a definition of an "analytic space" is given at the very end of Scholze's two semester lecture course [Sch1, Sch2] in Bonn. But why bother and not treat $\mathbb{R}$ and $\mathbb{Q}_p$ simply with different techniques?

The following is of course highly (!) speculative: Such a theory of analytic spaces might support a reasonable notion of shtukas over $\mathbb{Z}$ whose moduli spaces would vastly generalize Shimura varieties. Furthermore, in the cohomology of such spaces one might speculate to find for example non-regular algebraic automorphic forms (e.g. Maass forms on the complex upper half plane with Laplacian eigenvalue $\frac{1}{4}$) which do not show up in the cohomology of locally symmetric spaces. Thus making their study accessible to the tools of algebraic geometry.

Contents of this seminar. Our plan is as follows:

- Talks 1–6: Language of condensed mathematics.
- Talks 7–10: Solid modules and applications (=theory over $\mathbb{Q}_p$, see below).
- Talks 11–12: Liquid modules (=theory over $\mathbb{R}$).

Talks 1–6 are needed for both talks 7–10 and talks 11–12. However, the latter two series of talks can be understood (almost) independently of each other.

Concerning difficulty, talks 1–7 and 11 are suitable for good Master students and PhD students. Talks 8–9 require some more familiarity with derived categories, and Talk 10 additionally[1] with the functor formalism for coherent cohomology in algebraic geometry. Talk 12 requires familiarity with a few notions from functional analysis.

Going into more detail the first series of talks is devoted to learning the basics of condensed mathematics and roughly covers [Sch1, Lectures 1–4]. The second series of talks covers [Sch1, Lectures 5–8] and might be regarded as geometry over $\mathbb{Q}_p$, or more widely as adic geometry understood in a broad sense including algebraic geometry[2]. A highlight will be the construction of a coherent duality, generalizing Serre and Grothendieck duality. The last two talks deal with the theory over $\mathbb{R}$ where the reference is [Sch2]. This involves classical tools from functional analysis such as Banach and Smith spaces. For simplicity, we will only glimpse at the theory over $\mathbb{R}$ and will not discuss

[1] It might also be helpful, though not strictly necessary, to know some adic spaces.

[2] Just as the category of schemes is contained fully faithfully in the category of adic spaces.

globalizations of the aforementioned constructions in this seminar. That is, we will not explain how to "glue" to obtain the actual category of analytic spaces (just as manifolds are glued from affine spaces). In practice, this means skipping [Sch1, Lectures 9–11] and in fact most of [Sch2].

Getting started. It is highly recommended to read the introductions of [Sch1, Sch2]. Also Scholze's talk at MSRI gives an excellent overview of the contents of Talks 1–10 of this seminar. As the theory of condensed mathematics is brand new, the only sources are in fact Scholze's lecture notes. All participants –especially the speakers– are explicitly encouraged to contact one of the organizers concerning questions or additional textbooks on standard terms used in our primary source.

At a glance.

- Topic: Learning seminar on condensed mathematics.
- Audience: Most talks can be given by good Master students or PhD students.
- Main Sources: [Sch1, Sch2], Scholze's talk at MSRI
- Meetings: 12.11, 26.11, 03.12, 17.12, 14.01, 28.01
- Program discussion for the next term: 11.02

Prerequisites and Remarks.

- Talks 1–6: Categorical language (functors, adjoint functors, (co)limits, right (left) adjoint functors commute with (co)limits[3]); Homological algebra (additive/abelian categories, complexes, five lemma, snake lemma).
- Talk 4: Grothendieck sites and their sheaves. It will be helpful to know this for other talks as well, although it is not strictly necessary. Spectral sequences.
- Talks 3–12: Derived categories and derived functors as covered last term, see [DaFra-SS20].
- Throughout the seminar we *ignore* set-theoretic problems/choice of uncountable strong limit cardinals/etc.

List of Talks

In the following is a detailed list of talks according to our six meetings, each having two talks. Every talk is 60 minutes long. We encourage all participants to ask many questions!

1. Leitfaden & Getting condensed, 12.11.

Leitfaden. This is a short introductory lecture (ca. 15 min) to our actual program, given by one of the organizers. All speakers are encouraged to refer to this Leitfaden at the beginning of their talk. For a more detailed introduction, we refer to Scholze's MSRI talk.

Talk 1: The condensed world [Sch1, 1, 2]*,* [Stacks, 08ZW]*.*

(1) Briefly recall that a profinite space is a compact totally disconnected space or, equivalently, the filtered projective limit of finite discrete spaces [Stacks, 08ZW].
(2) Definition of the profinite site (it is the pro-étale site of a geometric point, but there is no need to elaborate on that – or even mention it). Define Condensed sets/groups/rings/... Spell out the notion of sheaf in this setting [Sch1, 1.2].
(3) Recall the definition of a compact extremally disconnected space ([Sch1, 2.4]) and explain that every compact Hausdorff space has a covering by a compact extremally disconnected space (in particular recall the Stone-Čech compactification). Show the equivalent definitions of condensed sets as sheaves on the compact Hausdorff site ([Sch1, 2.3]) or on the site of extremally disconnected spaces ([Sch1, 2.7]).

[3] One of the organizers (Torsten) has a brisk 10 page introduction to this topic which he distributes in his lecture when he assumes that everybody has heard of a category. Please feel free to contact him.

Talk 2: Topological spaces as condensed sets [Sch1, 1], [Str, 3].

(1) Recall definition of compactly generated spaces, give some examples (first countable or Hausdorff locally compact spaces are compactly generated). Show that if $X_1 \to X_2 \to \dots$ is a sequence of closed embeddings of compactly generated (weakly) Hausdorff spaces, then $\mathrm{colim}_n X_n$ is compactly generated ([Str, 3.6]).
(2) Define the functor $X \mapsto \underline{X}$ from top spaces/groups/... to condensed sets/groups/... and show that it is (fully) faithful on the category of (compactly generated) topological spaces ([Sch1, 1.7]). Show that it has a left adjoint.
(3) Show that via the functor $X \mapsto \underline{X}$ compact Hausdorff spaces correspond to quasi-compact quasi-separated condensed sets ([Sch2, 1.2]). Elaborate also on [Sch2, 1.2 (4)] using [Str, 3.8] instead of the reference to Bhatt-Scholze.

2. Condensed abelian groups and their cohomology I, 26.11.

Talk 3: The category of condensed abelian groups [Sch1, 2], [Sch2, 2], [Stacks, 079B], [Wd].

(1) Explain the functor $T \mapsto \mathbb{Z}[T]$ from the category of condensed sets to the category of condensed abelian groups (left adjoint to the forgetful functor). For S a profinite space describe $\mathbb{Z}[S]$ as in [Sch2, 2.1] and deduce that $\mathbb{Z}[S]$ is the condensed abelian group attached to an ind-compact topological abelian group ([Sch2, 2.2]).
(2) Discuss Example [Sch1, 1.9].
(3) Proof the properties of the category of condensed abelian groups ([Sch1, 2.2 ff], [Stacks, 079B] for an elaboration on Grothendieck's AB conditions):
   (i) It is abelian,
   (ii) limits and colimits can be calculated as for presheaves if restricted to the site of compact extremally disconected spaces (hence limits and colimits behave as for abelian groups),
   (iii) it is generated by compact projective objects $\mathbb{Z}[S]$ for S compact extremally disconnected (explain what this means),
   (iv) it has an associative, symmetric tensor product, and tensoring with $\mathbb{Z}[T]$ is exact for every condensed set T,
   (v) it has an internal $\underline{\mathrm{Hom}}$.
(4) We obtain the derived category $D(\mathrm{Cond(Ab)})$ of the abelian category of condensed abelian groups. Remark that right derivations and left derivations of all additive functor $\mathrm{Cond(Ab)} \to \mathcal{A}$, $\mathcal{A}$ any abelian category, exist (this follows from many standard results combined, see e.g. [Wd]). As examples for derived functors mention $R\Gamma(S, -)$ for S a compact Hausdorff space (and set $H^i(S, -) := R^i\Gamma(S, -)$), $R\underline{\mathrm{Hom}}(-, -)$ and $- \otimes^L -$.

Talk 4: Digression: Computing cohomology via simplicial hypercovers [Stacks]. The goal of the talk is to explain the general formalism that explains why cohomology on compact sets X can be computed by hypercoverings of X by profinite spaces.

(1) Recall the notion of simplicial objects of a category $\mathcal{C}$ ([Stacks, 016A]), the n-skeleton of a simplicial object ([Stacks, 017Z]), and the n-coskeleton as the right adjoint to the n-skeleton ([Stacks, 0AMA]). Remark that the n-coskeleton exists if $\mathcal{C}$ is finitely complete ([Stacks, 0183]). If time permits, make the construction of the 0-th coskeleton concrete.
(2) Recall the notion of a hypercovering of an object in a finitely complete site $\mathcal{C}$ but only as a simplicial object of $\mathcal{C}$[4], see [Stacks, 01G5]. Give example [Stacks, 01G6] but again only with a covering consisting of a single morphism.
(3) Define for an abelian presheaf on a site $\mathcal{C}$ its Čech cohomology with respect to a simplicial object of $\mathcal{C}$ (and in particular w.r.t. to a hypercovering) ([Stacks, 01GU]). For this recall the Dold-Kan correspondence for cosimplicial objects ([Stacks, 019H]). Remark that this

[4] i.e., all semi-representable objects are families consisting of a single object which suffices if one can pass from a family to a single object by taking the disjoint union (= coproduct) of the objects in a semi-representable object. For instance for the site of compact spaces all coverings are finite by definition and one can replace a finite family of compact spaces by its disjoint union.

generalizes the usual Čech cohomology and that $\check{H}^0$ computes global sections for a sheaf and for a hypercovering ([Stacks, 01GV]).

(4) Recall the spectral sequence that links Čech cohomology w.r.t. a hypercovering with usual cohomology ([Stacks, 01GY]).

(5) Consider the special case that $\mathscr{F}$ is an abelian sheaf on the site that K is a hypercovering of an object X consisting of objects K_n such that $H^q(K_n, \mathscr{F}) = 0$ for all n and for all $q \neq 0$. Then the spectral sequence degenerates and $H^p(X, \mathscr{F})$ is the cohomology of the complex $\Gamma(K_0, \mathscr{F}) \to \Gamma(K_1, \mathscr{F}) \to \dots$ associated to the cosimplicial abelian group $\Gamma(K, \mathscr{F})$ by Dold-Kan correspondence.

(6) Remark that for every profinite space S one has $H^p(S, \mathscr{F}) = 0$ for all abelian sheaves $\mathscr{F}$ and all $p > 0$ ([Stacks, 0A3F]). Hence cohomology on compact sets X can be computed by hypercoverings of X by profinite spaces.

3. Condensed abelian groups and their cohomology II, 03.12.

Talk 5: Computation of the cohomology of $\mathbb{Z}$ *and of* $\mathbb{R}$ [Sch1, 3].

(1) Recall (without proof) the lemma that $H^i_{\text{sheaf}}(S, \mathbb{Z})$ commutes with filtered projective limits in S (proof of [Sch1, 3.1]). Do not mention Čech cohomology which we identify with sheaf cohomology on (para)compact Hausdorff spaces.

(2) Prove that $H^i(S, \mathbb{Z})$ coincides with usual cohomology $H^i_{\text{sheaf}}(S, \mathbb{Z})$ of the locally constant sheaf $\mathbb{Z}$ on the compact Hausdorff space S ([Sch1, 3.2]).

(3) Show that $H^i(S, \mathbb{R}) = 0$ for all $i > 0$ and that $H^0(S, \mathbb{R}) = C(S, \mathbb{R})$ ([Sch1, 3.3]). Remark that the same proof also works if $\mathbb{R}$ is replaced by a real Banach space.

Talk 6: Locally compact abelian groups as condensed abelian groups [Sch1, 4].

(1) Recall the structure result on locally compact abelian groups and in particular the compact open topology ([Sch1, 4.1]).

(2) Prove [Sch1, 4.2].

(3) State and sketch the proof of [Sch1, 4.3]. For this use [Sch1, 4.5] without proof (but mention that it works for an abelian group object in any topos) and deduce [Sch1, 4.8].

(4) Explain, why [Sch1, 4.3] allows to compute $R\underline{\mathrm{Hom}}$ between locally compact abelian groups.

4. Return of the Solids, 17.12. It is recommended to prepare the following two talks together. The principal aim is to understand the statement and proof of [Sch1, 5.8] and the computations in [Sch1, 6.4].

Talk 7: Solids I [Sch1, 5].

(1) Motivate solid rings/modules [Sch1, beginning of 5] and [Sch2, 2]. It might be good to mention one of the examples [Sch1, 6.4] as well.

(2) Introduce solid abelian groups and solid abelian complexes [Sch1, 5.1]; take [Sch1, 5.2, 5.3] into account. Explain the relation between solid abelian groups and measures: $\mathbb{Z}[S]^{\blacksquare} = \underline{\mathrm{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})$.

(3) State the theorem of Nöbeling [Sch1, 5.4] and explain how to construct a basis (without proof which uses transfinite induction); deduce [Sch1, 5.5].

(4) State and prove [Sch1, 5.6, 5.7]; the arguments are a nice application of what we have learned so far.

(5) Finish by stating [Sch1, 5.8].

Talk 8: Solids II [Sch1, 5, 6].

(1) Give an overview of the proof (say at most 20-25 min) of Theorem 5.8 ([Sch1, 5.9, 5.10, beginning of Lecture 6]): What is "formal"? What are key computations? What is the interrelation to other theorems proven in this seminar? We will not be able to see the full proof in detail. Your lecture however should help the audience to access the arguments more easily.

(2) State [Sch1, 6.1] and sketch its proof.
(3) Introduce the tensor product $\otimes^{\blacksquare}$ and its derived variant $\otimes^{L\blacksquare}$ as in [Sch1, 6.2] without proof; explain the sample computations [Sch1, 6.3, 6.4] and also [Sch1, 6.5] if time permits.

5. Coherent duality, 14.01. In these talks we give an application of the theory developed so far to the construction of coherent duality, generalizing Serre and Grothendieck duality. For simplicity we restrict ourself to the affine case [Sch1, 7, 8], skipping the globalization [Sch1, 9–11].

Talk 9: Analytic rings [Sch1, 7]*,* [Sch2, 6]*.* We want to extend the results in Talks 7–8 from abelian groups to A-modules, for some ring A. This leads to (pre-)analytic rings.

(1) Introduce pre-analytic rings [Sch1, 7.1] and maps of pre-analytic rings [Sch1, above 7.7]; give [Sch1, 7.3]. Observe that we have maps $\mathbb{Z}_{\blacksquare} \to (A, \mathbb{Z})_{\blacksquare} \to A_{\blacksquare}$. Also interprete $\mathbb{Z}_{p,\blacksquare} \to (\mathbb{Q}_p, \mathbb{Z}_p)_{\blacksquare} \to \mathbb{Q}_{p,\blacksquare}$ in terms of measures, see below [Sch1, 7.10].
(2) State [Sch1, 7.2] as lemma[5] and introduce analytic rings [Sch1, 7.4], see also [Sch2, 6.12] for an equivalent definition. Define maps of analytic rings [Sch1, above 7.7] and state [Sch1, 7.14] without proof.
(3) For an analytic ring $\mathcal{A}$ introduce the notation $\mathrm{Mod}_{\mathcal{A}}^{\mathrm{cond}}$ and $D(\mathcal{A}) := D(\mathrm{Mod}_{\mathcal{A}}^{\mathrm{cond}})$. State and prove [Sch1, 7.5, 7.7]; have a look at [Sch2, 6.15] as well.
(4) State and prove [Sch1, 7.9]; mention that $A_{\blacksquare}$ is analytic (proven in the next talk) for a finitely generated $\mathbb{Z}$-algebra A; conclude that $\mathbb{Z}_{\blacksquare} \to (A, \mathbb{Z})_{\blacksquare} \to A_{\blacksquare}$ are maps of analytic rings.

Talk 10: Local Serre duality [Sch1, 8, 9, 11]*.* For a finitely generated $\mathbb{Z}$-algebra A, we use the maps $\mathbb{Z}_{\blacksquare} \to (A, \mathbb{Z})_{\blacksquare} \to A_{\blacksquare}$ to construct cohomology with compact support $f_! \colon D(A_{\blacksquare}) \to D(\mathbb{Z}_{\blacksquare})$ where $f \colon \mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{Z})$.

(1) Motivate that the map $(A, \mathbb{Z})_{\blacksquare} \to A_{\blacksquare}$ might be seen as a compactification[6], see [Sch1, 9.6]. It is good to mention valuation spectra, but avoid the use of general adic spaces.
(2) State [Sch1, 8.1, 8.2]; explain local Serre duality [Sch1, 8.4] and the formula for the dualizing complex [Sch1, 8.5]. Then follow the outline in the rest of Lecture 8 as time permits. In particular, restrict to the case $A = \mathbb{Z}[T]$ and skip the appendix to Lecture 8.
(3) Finish by mentioning that this construction can globalized and then supports a full six functor formalism for coherent cohomology [Sch1, 11].

6. Liquids strike back, 28.01.

Talk 11: Banach and Smith spaces I [Sch2, 3, 4]*.*

(1) Recall basic notions of real analysis ([Sch2, 3.2ff]):
   (i) (complete) locally convex vector spaces, every Banach space is complete locally convex (see also [Bou-TVS, II, §4.1]),
   (ii) absolutely convex subsets
(2) Define the topological vector space $\mathcal{M}(S)$ of ("signed Radon") measures on a profinite set S ([Sch2, 3.2ff]). Remark that it is compactly generated as countable union of compact spaces endowed with the colimit topology (see Talk 2).
(3) The condensed $\mathbb{R}$-vector space attached to real complete locally convex spaces ([Sch2, 3.4]).
(4) Define $\mathcal{M}$-complete condensed $\mathbb{R}$-spaces and Smith space ([Sch2, 4.1, 4.2]). Show that any Smith space is a topological vector space (more precisely of the form $\underline{V}$, where V is a topological vector space that is compactly generated). Show that $\mathcal{M}(S)$ is a Smith space.

[5] All our rings are "discrete"; an example is $\underline{A}[S] = \mathbb{Z}[S]$ with its natural map to $\mathcal{A}[S] = \mathbb{Z}[S]^{\blacksquare}$.
[6] Let $\tilde{\mathbb{Z}}$ denote the integral closure of $\mathbb{Z}$ in A. Then $(A, \tilde{\mathbb{Z}})_{\blacksquare} = (A, \mathbb{Z})_{\blacksquare}$, $(A, \tilde{\mathbb{Z}})$ is a Huber pair and $\mathrm{Spa}(A, \tilde{\mathbb{Z}}) = \mathrm{Spv}(A)$ is proper over $\mathbb{Z}$ (in the sense of adic spaces). The space $\mathrm{Spa}(A, A)$ embeds in $\mathrm{Spv}(A)$ as the "unit disc", i.e., those valuations taking values ≤ 1 on A.

Talk 12: Banach and Smith spaces II [Sch2, 4, 6].

(1) Prove the duality between Smith spaces and Banach spaces ([Sch2, 4.7]).
(2) Define the tensor product of $\mathcal{M}$-complete condensed $\mathbb{R}$-vector spaces ([Sch2, 4.9]). If time permits relate this construction to Grothendieck's projective and injective tensor product.
(3) Define $\mathcal{M}_p(S)$ for $0 < p \leq 1$ and S profinite ([Sch2, 6.3]).
(4) State the Main Theorem [Sch2, 6.5] about the category of p-liquid condensed spaces for $0 < p \leq 1$. No proof.

7. Program discussion, 11.02. After our last meeting we will send a reminder to collect possible topics for the next term in advance. Of course, everybody is welcome to suggest a topic during the discussion spontaneously – everything goes! Then we collect the topics, discuss and vote.

References

[Bou-TVS] N. Bourbaki, *Topological Vector Spaces*, Chapters 1–5, Springer (1987). 5

[DaFra-SS20] DaFra seminar on Derived Categories, summer term 2020, program available at https://www.mathematik.tu-darmstadt.de/media/algebra/dafra/ 2

[Sch1] P. Scholze: *Lectures on Condensed Mathematics*, lecture notes available at https://www.math.uni-bonn.de/people/scholze/. 1, 2, 3, 4, 5

[Sch2] P. Scholze: *Lectures on Analytic Geometry*, lecture notes available at https://www.math.uni-bonn.de/people/scholze/. 1, 2, 3, 4, 5, 6

[Stacks] The Stacks project, https://stacks.math.columbia.edu. 2, 3, 4

[Str] N.P. Strickland, *The Category of CGWH Spaces*, lecture notes available at https://neil-strickland.staff.shef.ac.uk/courses/homotopy/. 3

[Wd] Private communication to T. Wedhorn. 3

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