Part 2. Continuous mappings

1. THE EXPONENTIAL MAP AND OTHER EXAMPLES OF MAPPINGS

1.1. **Mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$.** With this section we begin the study of mappings, which is the key topic of analysis. Mappings were defined in I1.1; here we present some examples.

In general, we consider maps $f\colon X \to \mathbb{R}^m$ where $X \subseteq \mathbb{R}^n$; we assume this unless stated otherwise. If the target dimension is one, $m = 1$, then we call the map a *function* or a scalar-valued function; when the target is complex, $m = 2$, we also consider the mapping a (complex-valued) function.

Let $X \subseteq \mathbb{R}^n$, and $f\colon X \to \mathbb{R}^m$. Then we call the set

$$\Gamma(f) := \big\{\big(x, f(x)\big) \mid x \in X\big\} \subseteq X \times \mathbb{R}^m \subseteq \mathbb{R}^{n+m}$$

the *graph* of f.

Examples. 1. The identity mapping

$$\mathrm{id}\colon X \to \mathbb{R}^n, \qquad \mathrm{id}(x) := x.$$

Its graph is the set $\{(x, x) \subseteq X \times \mathbb{R}^n \mid x \in X\} \subseteq \mathbb{R}^{2n}$.

2. We will later need the *floor function*

$$\lfloor . \rfloor\colon \mathbb{R} \to \mathbb{R}, \qquad \lfloor x \rfloor := \sup\{n \in \mathbb{Z} \mid n \leq x\};$$

the supremum exists by Thm. I,32. As always when $n = m = 1$, the graph is a subset of $\mathbb{R}^2$ and can be sketched easily.

3. *Linear mappings*,

$$L\colon \mathbb{R}^n \to \mathbb{R}^m, \qquad L(x) := Ax,$$

where $A = (a_{ij})$ is an $m \times n$-matrix. For $n = 2$, $m = 1$, the graph is a plane through 0. Let us now consider the case $n = m = 2$. We can visualize it by picturing the image of a grid. Linearity means that each grid square of the preimage is mapped onto an image parallelogramme. All these parallelogrammes are congruent, and the image of the unit spare has area $\det L$. If the map has only rank 1 then the image grid collapses onto a line.

4. *Complex functions* $f\colon \mathbb{C} \to \mathbb{C}$ can also be visualised by depicting the image of a square grid. Simple algebraic operations give rise to interesting complex functions: For instance, $z \mapsto z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$ doubles the polar angle, and squares the modulus of z.

5. The case $n = m$ is often interpreted in terms of *vector fields*: The mapping attaches to each point x of the domain the vector $f(x)$ (speed of wind in space, etc.)

Only $\mathbb{R}$ can be ordered. Functions which respect the order are particularly useful:

Definition. Let $D \subseteq \mathbb{R}$, and $f\colon D \to \mathbb{R}$ a function. Then f is *(monotonously)* $\left\{ \substack{\textit{increasing} \\ \textit{decreasing}} \right\}$ if for each pair $x, y \in D$ with $x < y$ there holds $\left\{ \substack{f(x) \le f(y) \\ f(x) \ge f(y)} \right\}$. When the inequality is strict, the function is *strictly* monotone.

Examples. 1. $f\colon \mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\} \to \mathbb{R}$, $f(x) = x^2$ is strictly increasing, since $0 \le x < y$ implies $x^2 \le xy < y^2$.

2. Similarly, x^n is an increasing function from $\mathbb{R}_0^+$ to $\mathbb{R}$ for each $n \in \mathbb{N}$; for n odd, this holds on all of $\mathbb{R}$.

A strictly monotone function $f\colon D \to \mathbb{R}$ is injective, and hence has an *inverse function [Umkehrfunktion]* $f^{-1}\colon f(D) \to \mathbb{R}$; it can be defined by $f^{-1} \circ f = \mathrm{id}$ on D.

1.2. **The exponential series.** We want to introduce the most famous mathematical function which is not elementary. We will work directly in the complex setting; if you feel uncomfortable with this choice, assume on a first reading that z is a real number.

For $z \in \mathbb{C}$ let us define the *exponential series* (Newton 1669 for $\mathbb{R}$)

$$\exp(z) := 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Note that $\exp(x) \in \mathbb{R}$ for real x; indeed, then all partial sums have vanishing imaginary part.

By the next theorem, the complex exponential series is a function from $\mathbb{C}$ to $\mathbb{C}$, which we will later use to introduce the trigonometric functions $\sin, \cos, \ldots$, as well as the number π.

Theorem 1. *The series* $\exp(z)$ *converges for each* $z \in \mathbb{C}$*, that is,* $\exp$ *defines a function from* $\mathbb{C}$ *to* $\mathbb{C}$*.*

Proof. For $z = 0$ there is nothing to show. Now fix $z \neq 0$. The ratio test gives for the k-th term $a_k := \frac{z^k}{k!}$

$$\frac{|a_{k+1}|}{|a_k|} = \frac{|z^{k+1}|}{(k+1)!} \cdot \frac{k!}{|z^k|} = \frac{|z|}{k+1}.$$

Hence if we choose $N \in \mathbb{N}$ depending on z such that $|z| \le \frac{N}{2}$, we obtain for all $k \geq N$

$$\frac{|a_{k+1}|}{|a_k|} \le \frac{N}{2(k+1)} \le \frac{1}{2} \frac{k}{k+1} < \frac{1}{2}.$$

Therefore, the exponential series passes the ratio test with $q = \frac{1}{2}$. □

Particular values: $\exp(0) = 1$ and $\exp(1) = 2.71828\ldots =: e$ is the *Euler number* (Euler 1736).

Many properties of the exponential function will be derived from the following error bound on the partial sums of $\exp(z)$:

Theorem 2 (Remainder term estimate). *For $k \in \mathbb{N}_0$, let $R_k(z)$ be the series defined by*

$$R_k(z) := \frac{1}{k!}z^k + \frac{1}{(k+1)!}z^{k+1} + \ldots = \sum_{j=k}^{\infty} \frac{z^j}{j!}. \tag{1}$$

Then

$$\bigl|R_k(z)\bigr| \leq 2\frac{|z|^k}{k!} \qquad \text{for} \qquad |z| \leq \frac{k+1}{2}.$$

We can write $\exp(z) = s_{k-1}(z) + R_k(z)$. Here s_{k-1} is a polynomial, namely the $(k-1)$-st partial sum, and $R_k(z)$ measures the error of approximating exp by this polynomial. In view of the growth properties we establish in 4.3 below, it is no surprise that the error estimate only holds on the bounded sets $B_{(k+1)/2} \subseteq \mathbb{C}$.

Proof. We apply the infinite triangle-inequality I(46) to the series (1), and factor out $\frac{|z|^k}{k!}$. This gives

$$\begin{aligned}\bigl|R_k(z)\bigr| &\leq \frac{|z|^k}{k!}\left[1 + \frac{|z|}{k+1} + \frac{|z|^2}{(k+1)(k+2)} + \ldots\right] \\ &\leq \frac{|z|^k}{k!}\left[1 + \frac{|z|}{k+1} + \Bigl(\frac{|z|}{k+1}\Bigr)^2 + \ldots\right].\end{aligned}$$

Let us now apply the comparison test with the geometric series, this time in a quanitative way: Using our assumption $|z| \leq \frac{k+1}{2}$ we can majorize $[\ldots]$ with the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2$ (that is, we invoke Corollary I,47). This establishes the desired error estimate. ☐

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1.3. **Growth laws.** We want to explain the exponential function in an off-hand way using derivatives. The two simplest growth laws for a quantity $f(t)$ which depends on the time t are:

1. *Additive growth*, where f is the diameter of a tree, the height of sediment deposits, or interest gain of a constant capital. In this case, $f' = c$ with a constant $c \in \mathbb{R}$, and so the growth law is linear: $f(t) = ct + a$ where $a, c \in \mathbb{R}$.

2. *Multiplicative growth*, where f is the size of population, an investment (growing with fixed interest rates), or radioactive material (decaying with radiation). Here, the growth is proportional to the quantity $f(t)$, that is $f' = cf$ or $\frac{f'}{f} = c$ with a constant $c \in \mathbb{R}$. We claim the growth law is exponential, $f(t) = a\exp(ct)$ mit $a, c \in \mathbb{R}$. Indeed, differentiating the exponential series term by term we obtain

$$\bigl(\exp(x)\bigr)' = \Bigl(\sum_{k=0}^{\infty}\frac{1}{k!}x^k\Bigr)' \overset{(*)}{=} \sum_{k=0}^{\infty}\Bigl(\frac{1}{k!}x^k\Bigr)' = \sum_{k=1}^{\infty}\frac{k}{k!}x^{k-1} = \sum_{k=1}^{\infty}\frac{1}{(k-1)!}x^{k-1} = \exp(x),$$

and similarly $a\exp(cx)' = c\big(a\exp(cx)\big)$. Finite sums can be differentiated termwise. However, for infinite sums, the step $(*)$ needs justification, which we will provide later.

Are there any other solutions to the differential equation $f' = cf$? Perhaps, elementary functions f solve, too? This is not the case: Suppose f is a polynomial $p(x) = a_0 + \ldots + a_k x^k$; if $a_k \neq 0$ we call k the *degree* of p. Note that if p has degree $k \geq 1$, then p' has degree $n-1$. Thus $p' = cp$ (with $c \neq 0$) can hold only if $p \equiv 0$. This is but one example; but soon we will see that the differential equation $f' = cf$ has a unique solution for each given initial value $f(0) = a$, namely only $f(x) := a\exp(cx)$.

Consequently, the modelling of the simple growth law $f' = cf$ requires the study of an infinite series $\sum \frac{1}{k!}x^k$, that is, a function which can only evaluated by approximation. Likewise, $1, \cos x, \sin x$ are the edge lengths of a right-angled triangle (with angle x); again, $\cos x$ and $\sin x$ are series.

So the completion process leading from rational to real numbers has an analogue when dealing with functions: We must "complete" the polynomials to the class of infinite series $\lim_{k\to\infty} p_k(x) = \sum_{k=0}^{\infty} a_k x^k$ if we want to solve natural differential equations.

1.4. **The functional equation for** exp. By definition, we have $a^{17} \cdot a^3 = a^{17+3}$ or, in general, $a^k a^l = a^{k+l}$ for $k, l \in \mathbb{N}$. Our goal is to assert the same property for the exponential function, that is, $\exp(z)\exp(w) = \exp(z+w)$. Eventually, we will use this to write $\exp(z)$ in the form e^z. Since $\exp(z)\exp(w)$ is a product of two series, we will now consider the product of two complex series in general.

When we multiply finite sums,

$$(a_0 + \ldots + a_k)(b_0 + \ldots + b_l),$$

we obtain $(k+1)\cdot(l+1)$ terms; when we sum them, their order is irrelevant. However, when we multiply two series $\sum_{j=0}^{\infty} a_j \ \sum_{j=0}^{\infty} b_j$ of complex numbers, we need to add the infintitely many terms listed in the following table:

$$\begin{array}{lllll}
a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \ldots \\
a_1b_0 & a_1b_1 & a_1b_2 & \ldots & \\
a_2b_0 & a_2b_1 & \ldots & & \\
a_3b_0 & \ldots & & & \\
\ldots & & & &
\end{array}$$

Remember from the alternating harmonic series that the order of summation does matter when dealing with infinitely many terms. Our approach here is to consider a particular, convenient way of adding all the terms, namely via enumeration along diagonals. That is, we first take diagonal sums,

$$c_k := a_k b_0 + \ldots + a_0 b_k \qquad \text{for } k \in \mathbb{N}, \tag{2}$$

and then add the c_k. Assuming absolute convergence, we obtain the desired result:

Lemma 3 (Cauchy-product of series)**.** *Let* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=0}^{\infty} b_k$ *be absolutely convergent complex series, and* c_k *as in* (2). *Then* $\sum_{k=0}^{\infty} c_k$ *is also absolutely convergent, and*

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \sum_{k=0}^{\infty} b_k.$$

Without the assumption of absolute convergence this statement is false: In general the infinite sum over all terms $a_i b_j$ will depend on the order of summation chosen!

Proof. We denote the partial sums by

$$s_k := a_0 + \ldots + a_k \to s, \qquad t_k := b_0 + \ldots + b_k \to t, \qquad u_k := c_0 + \ldots + c_k.$$

Let us first show that $\sum c_k$ converges to st. This means we want to prove that $st - u_k \to 0$ as $k \to \infty$. As the complex sequences s_k and t_k converges, their product converges as well (by the unchanged proof of Thm. I,16(*iii*)). Hence $s_k t_k \to st$ as $k \to \infty$ and so it suffices to show $s_k t_k - u_k \to 0$ as $k \to \infty$. Written in this form, the claim involves only finite sums.

The expression $s_k t_k - u_k$ contains the products $a_i b_j$ for certain pairs of indices. To be explicit, let $\Box_k := \{(i,j) \in \mathbb{N}_0^2 \mid 0 \le i,j \le k\}$ be a square of indices, and $\Delta_k := \{(i,j) \in \mathbb{N}_0^2 \mid i + j \le k\}$ be the lower triangle contained in $\Box_k$. We can now write

$$s_k t_k - u_k = \sum_{(i,j) \in \Box_k \setminus \Delta_k} a_i b_j.$$

Since $\Box_{\lfloor k/2 \rfloor} \subseteq \Delta_k$ for all $k \in \mathbb{N}_0$ we have that $\Box_k \setminus \Delta_k \subseteq \Box_k \setminus \Box_{\lfloor k/2 \rfloor}$ for all k. This gives

$$\left| s_k t_k - u_k \right| = \left| \sum_{(i,j) \in \Box_k \setminus \Delta_k} a_i b_j \right| \overset{\Delta\text{-inequ.}}{\le} \sum_{(i,j) \in \Box_k \setminus \Delta_k} |a_i||b_j| \le \sum_{(i,j) \in \Box_k \setminus \Box_{\lfloor k/2 \rfloor}} |a_i||b_j|.$$

Let us now use the assumption that $\sum a_k$ and $\sum b_k$ are absolutely convergent. This means the partial sums,

$$A_k := |a_1| + \ldots + |a_k| \qquad \text{and} \qquad B_k := |b_1| + \ldots + |b_k|,$$

are convergent sequences. Consequently, the product sequence $(A_k B_k)_{k \in \mathbb{N}}$ is also convergent and, in particular, a Cauchy sequence (Thm. I,27), so that

$$\left| A_k B_k - A_{\lfloor k/2 \rfloor} B_{\lfloor k/2 \rfloor} \right| \to 0 \qquad \text{as } k \to \infty.$$

The claim follows from combining our results:

$$\left|s_k t_k - u_k\right| \le \sum_{(i,j)\in\square_k\setminus\square_{\lfloor k/2\rfloor}} |a_i||b_j| = \left|A_k B_k - A_{\lfloor k/2\rfloor} B_{\lfloor k/2\rfloor}\right| \to 0 \qquad \text{as } k\to\infty$$

It remains to show that $\sum c_k$ is absolutely convergent. We apply what we have proven so far to the product of the two real series $\sum |a_k|$ and $\sum |b_k|$. On the one hand we find that $\sum |a_k| \sum |b_k| = \sum_{k=0}^{\infty}\big(\sum_{j=0}^{k} |a_j||b_{k-j}|\big)$ and the right hand side converges. On the other hand, the triangle inequality gives $|c_k| \le \sum_{j=0}^{k} |a_j||b_{k-j}|$. Taken together, this proves that $\big(\sum_{j=0}^{k} |a_j||b_{k-j}|\big)_{k\in\mathbb{N}}$ is a convergent sequence, majorizing $|c_k|$. By Corollary I,47 the series $\sum |c_k|$ converges, that is, $\sum c_k$ is absolutely convergent. ☐

Let us now apply the lemma to the exponential series.

Theorem 4. *For all* $z, w \in \mathbb{C}$ *we have*

$$\exp(z+w) = \exp(z)\exp(w). \tag{3}$$

Proof. Since exp is absolutely convergent we can evaluate the left hand side of (3) by the Cauchy product of $\sum a_k := \sum \frac{z^k}{k!}$ with $\sum b_k := \sum \frac{w^k}{k!}$. We invoke the Binomial Theorem to compute the coefficients

$$c_k := \sum_{j=0}^{k} \frac{z^j}{j!}\cdot\frac{w^{k-j}}{(k-j)!} = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} z^j w^{k-j} \overset{\text{Binomial thm. I,5}}{=} \frac{1}{k!}(z+w)^k \quad \text{for all } k\in\mathbb{N}_0.$$

Thus the Cauchy product is

$$\exp(z)\exp(w) = \sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty}\frac{1}{k!}(z+w)^k = \exp(z+w).$$

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In particular we conclude $\exp(z)\exp(-z) = \exp(z-z) = \exp(0) = 1$, that is,

$$\exp(-z) = \frac{1}{\exp(z)}. \tag{4}$$

This has the following consequence for exp evaluated on real numbers:

Corollary 5. (i) $\exp(x) > 0$ *for all* $x\in\mathbb{R}$, *and* $\exp(x) > 1$ *for* $x > 0$.
(ii) $\exp\colon \mathbb{R}\to\mathbb{R}$ *is strictly increasing.*
(iii) *For* $k\in\mathbb{Z}$ *holds* $\exp(k) = e^k$, *where* $e\in\mathbb{R}$ *is the Euler number.*

Here, by definition $z^0 := 1$ for $z\in\mathbb{C}$ and $z^{-k} := \frac{1}{z^k}$ for $z\in\mathbb{C}-\{0\}$ and $k\in\mathbb{N}$.

Proof. (i) Let $x > 0$. Then $\exp(x) = 1 + x + \frac{1}{2!}x^2 + \ldots > 1$ and, using (4), also $\exp(-x) = \frac{1}{\exp(x)} > 0$.

(ii) For $x < y$ we use (i) to show

$$\exp(y) = \exp(y - x + x) \overset{(3)}{=} \underbrace{\exp(y-x)}_{>1} \underbrace{\exp(x)}_{>0} > \exp(x).$$

(iii) For $k = 0$, by definition $\exp(0) = e^0 = 1$ holds. Let now $k \in \mathbb{N}$. Using $\exp(1) = e$ and (3) gives

$$\exp(k) = \exp(1 + \ldots + 1) \overset{(3)}{=} \exp(1) \cdot \ldots \cdot \exp(1) = \exp(1)^k = e^k;$$

again, formally this is by induction. Finally,

$$\exp(-k) \overset{(4)}{=} \frac{1}{\exp(k)} = \frac{1}{e^k} = e^{-k}. \qquad \square$$

Property (iii) says that $\exp(x)$ interpolates the natural powers of e. Hence it makes sense to call exp the *exponential* function.

2. CONTINUITY IN TERMS OF SEQUENCES

We want to take the inverse function of $\exp\colon \mathbb{R} \to \mathbb{R}$, the logarithm log. An inverse function is defined on the range of the original function. But why does the range of exp consist of all positive real numbers? It is the continuity of exp which implies that the range has no gaps.

2.1. **Definition and examples.**

Historically, the significance of continuity was discovered only in the 19th century – long after the introduction of differentiation and integration. Before, continuity had been assumed tacitly, and similarly today most people will do the same, for instance, when interpreting the weather reading of Frankfurt airport for Darmstadt (what is the function in question?).

Definition (Bolzano 1817)**.** Let $X \subseteq \mathbb{R}^n$ and $f\colon X \to \mathbb{R}^m$. The mapping f is *continuous [stetig] at* $a \in X$, if

$$\lim_{k\to\infty} f(x_k) = f(a) \qquad \text{for each sequence } x_k \in X \text{ with } \lim_{k\to\infty} x_k = a. \tag{5}$$

The map f is *continuous*, if f is continuous in each $a \in X$.

Notation: We write

$$\lim_{x\to a} f(x) = c \quad :\Longleftrightarrow \quad \lim_{k\to\infty} f(x_k) = c \text{ for each sequence } x_k \in X \text{ with } x_k \to a.$$

This notation may also be used when $a \notin X$, for instance we could have $X := (a, b) \subseteq \mathbb{R}$.

Examples. 1. A constant function, $f(x) \equiv c \in \mathbb{R}^m$, is continuous: $\lim_{x\to a} f(x) = c = f(a)$
2. The identity mapping $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$, is continuous: $\lim_{x\to a} \mathrm{id}(x) = \lim_{x\to a} x = a = \mathrm{id}(a)$.
3. The floor function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \lfloor x \rfloor$, is not continuous at $a \in \mathbb{Z}$. Indeed, it is enough to find a particular sequence, such that (5) fails. We take $x_k := a - \frac{1}{k} \to a$:

$$\lim_{k\to\infty} \left\lfloor a - \frac{1}{k} \right\rfloor = \lim_{k\to\infty} (a-1) = a - 1 \quad \neq \quad \lfloor a \rfloor = a$$

On $\mathbb{R} - \mathbb{Z}$ the function is continuous (H11.2).

4. $f(x) := 1$ for $x \in \mathbb{Q}$ and $f(x) := 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ is nowhere continuous, but $xf(x)$ is continuous at 0 (P12.4).

5a) $\exp\colon \mathbb{C} \to \mathbb{C}$ is continuous at 0.
Proof: We use the Remainder Term Estimate, Theorem 2. With $k = 1$ it says that $R_1(z) = z + \frac{z^2}{2!} + \ldots = \exp(z) - 1$ is subject to $|R_1(z)| \le 2|z|$ when $|z| \le 1$. For each null sequence $(z_k)_{k\in\mathbb{N}}$ there exists $N \in \mathbb{N}$, such that $|z_k| \le 1$ for all $k \ge N$; then

$$|R_1(z_k)| = \big|\exp(z_k) - 1\big| \le 2|z_k| \to 0 \qquad \text{as } k \to \infty. \tag{6}$$

Therefore,

$$\lim_{z\to 0} \exp(z) = 1 = \exp(0),$$

which proves the continuity of exp in 0.
b) $\exp\colon \mathbb{C} \to \mathbb{C}$ is continuous.
Proof: We use a) together with the functional equation (3): For $z_k \to a$ as $k \to \infty$ we have

$$\lim_{k\to\infty} \exp(z_k) \overset{z_k = a + z_k - a}{=} \lim_{k\to\infty} \big(\exp(a)\exp(z_k - a)\big) = \exp(a) \lim_{k\to\infty} \exp(\underbrace{z_k - a}_{\to 0}) \overset{\text{a)}}{=} \exp(a).$$

6. The addition function

$$+\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \qquad (z,w) \mapsto z + w$$

is continuous.
Proof: Let $\lim_{k\to\infty}(z_k, w_k) = (z,w)$. By Thm. I,39($i$), equivalently $\lim_{k\to\infty} z_k = z$ and $\lim_{k\to\infty} w_k = w$. Therefore

$$\lim_{k\to\infty} (z_k + w_k) = \lim_{k\to\infty} z_k + \lim_{k\to\infty} w_k = z + w$$

Similarly, one can show that the product $z \cdot w$ is a continuous function $\cdot\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

2.2. **Operations which preserve continuity.** Various properties of continuous functions are direct consequences of the similar properties of limits:

Theorem 6. *Let* $\lambda \in \mathbb{R}$*. If* $f, g\colon X \to \mathbb{R}^n$ *are mappings which are continuous in* $a \in X$*, then also* $f+g$*,* λf*,* $\|f\|$ *are. The same holds for the product* fg *of two functions,* $f, g\colon X \to \mathbb{C}$ *and for the quotient* $\frac{f}{g}$*, where for the quotient we must assume that* $g\colon X \to \mathbb{C} \setminus \{0\}$*.*

Proof. The properties claimed follow from the respective laws for limits in $\mathbb{R}$ (Thm. I,27) or $\mathbb{R}^n$ (see p.46). As an example, let us show the first claim. Suppose that $x_k \to a$ as $k \to \infty$. Then

$$\begin{aligned} \big(f+g\big)(a) = f(a) + g(a) &= \lim_{k\to\infty} f(x_k) + \lim_{k\to\infty} g(x_k) \\ &= \lim_{k\to\infty} \big(f(x_k) + g(x_k)\big) = \lim_{k\to\infty} (f+g)(x_k), \end{aligned}$$

and so the limit $\lim_{x\to a}(f+g)(x)$ exists. Thus $f+g$ is continuous at a. □

Examples. 1. $z \mapsto z^k$ as a map from $\mathbb{C}$ to $\mathbb{C}$ is continuous for $k \in \mathbb{N}$. Indeed, the identity $z \mapsto z$ is continuous, and by the theorem also $z \mapsto z \cdot z$, etc. (induction).

2 *Polynomials* $p(z) = a_0 + a_1 z + \ldots + a_k z^k$, with $a_j \in \mathbb{C}$, are continuous on $\mathbb{C}$: Indeed, z^k is continuous by 1.; by the theorem each complex product function $a_j \cdot z^j$ is continuous; and so is the sum of these, again by the theorem.

A vector sequence converges iff all its components converge (Thm. I39(i)). Applying this to functions gives:

Theorem 7. *A vector-valued function* $f\colon X \to \mathbb{R}^m$ *is continuous at* $a \in X$ *if and only if all its component functions* $f_1\colon X \to \mathbb{R}$*, ...,* $f_m\colon X \to \mathbb{R}$ *are continuous at* a*.*

Example. A linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$,

$$L(x) = \big(a_{11}x_1 + \ldots + a_{1n}x_n,\ \ldots\ ,\ a_{m1}x_1 + \ldots + a_{mn}x_n\big) \qquad \text{where all } a_{ij} \in \mathbb{R},$$

is continuous. Indeed, since the identity mapping $x \mapsto (x_1, \ldots, x_n)$ is continuous, the theorem gives that its component functions x_j are continuous on $\mathbb{R}^n$. Using the rules for addition and multiplication, Thm. 6, we see that each component function $L_k(x)$, is continuous. By the theorem, the vector function L is continuous.

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Finally, we show that continuity is preserved under composition. This will allow us to derive the continuity of complicated functions from the known continuity of simple functions.

Theorem 8. *Let* $X \subseteq \mathbb{R}^n$ *and* $Y \subseteq \mathbb{R}^m$*. If* $f\colon X \to Y$ *is continuous at* a *and* $g\colon Y \to \mathbb{R}^l$ *is continuous at the point* $f(a)$*, then* $g \circ f\colon X \to \mathbb{R}^l$ *is continuous at* a*.*

Proof. Let $x_k \in X$ be an arbitrary sequence which converges to a. Since f is continuous we have $f(x_k) \to f(a)$. Feeding this sequence into the definition of continuity for g, we find $g\big(f(x_k)\big) = g\big(f(a)\big)$, as desired. $\square$

Example. Given a point $p \in \mathbb{R}^n$, the distance function to p is a continuous function, $\|x-p\|$. Indeed, we can consider it a composition of $g(y) := \|y\|$ with $f(x) := x - p$.

2.3. **Intermediate value theorem and invertibility of functions.** In this section, we consider functions f from intervals to $\mathbb{R}$.

A common explanation of continuity of functions is that their graph can be drawn in one turn. In extreme cases, this interpretation runs into problems: Perhaps, the drawing speed must be infinite! But the main content of the explanation is that the range does not leave any gaps—that is indeed true:

Theorem 9 (Intermediate value theorem, Bolzano 1817)**.** *Let f: $[a,b] \to \mathbb{R}$ be continuous.*
(i) *If $f(a) < 0$ and $f(b) > 0$ (or vice versa), then there exists (at least one) $x \in (a,b)$ with $f(x) = 0$, called a* zero *[Nullstelle]* of f.
(ii) *Similarly, each value c in between $f(a)$ and $f(b)$ is attained.*

On the other hand, $f\colon \mathbb{Q} \to \mathbb{Q}$, $f(x) := x^2 - 2$ has no zero! (Where does the following proof fail for that case?) So it is the completeness of $\mathbb{R}$ which is exploited essentially.

Proof. (i) We locate x by the method of interval bisection (compare with the proofs for the supremum, Thm. I,32, and Bolzano-Weierstrass, Thm. I,35).

We define nested intervals $(I_k := [a_k, b_k])_{k\in\mathbb{N}}$ with the property $f(a_k) < 0$ and $f(b_k) \geq 0$. So, recursively, we set $[a_1, b_1] := [a, b]$ and, for $k \geq 2$,

$$I_{k+1} := \begin{cases} [a_k, \frac{a_k+b_k}{2}] & \text{if } f(\frac{a_k+b_k}{2}) \geq 0, \\ [\frac{a_k+b_k}{2}, b_k] & \text{if } f(\frac{a_k+b_k}{2}) < 0. \end{cases}$$

Since (I_k) are nested intervals, there exists $x \in I_k$ for all k. By continuity we have

$$f(x) = \lim_{k\to\infty} f(a_k) \leq 0 \quad \text{and} \quad f(x) = \lim_{k\to\infty} f(b_k) \geq 0,$$

and so $f(x) = 0$ must hold.

(ii) Similar by replacing 0 with c in part (i). $\square$

Corollary 10. (i) $\exp\colon \mathbb{R} \to \mathbb{R}^+$ *is bijective.*
(ii) *For $k \in \mathbb{N}$ the power $x^k\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is bijective.*

Proof. (*i*) We asserted the continuity of exp before. Let $y \in \mathbb{R}^+$. We claim there exist numbers $k, l \in \mathbb{Z}$ with $\exp(k) = e^k < y < e^l = \exp(l)$. Indeed, as $e > 1$ we have $e^j \to \infty$ and $e^{-j} = \frac{1}{e^j} \to 0$ as $j \to \infty$, Therefore, by the intermediate value theorem, exp takes the value y. This shows exp is surjective to the positive reals. Moreover, by the strict monotonicity of exp asserted in Corollary 5(*ii*), exp is also injective.

(*ii*) The same arguments apply. The polynomial x^k is continuous (first example in 2.2). Moreover, $0^k = 0$ and $j^k \to \infty$ as $j \to \infty$. Again the intermediate value theorem proves the surjectivity of $x^k \colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$, while the strict monotonictity (see example in 1.1) proves the injectivity. □

2.4. **Open, closed, and compact subsets of $\mathbb{R}^n$.** In order to study properties of functions $f \colon X \to \mathbb{R}^m$ we need to single out domains $X \subseteq \mathbb{R}^n$ with good properties. The following kinds of subsets generalize the open and closed intervals of $\mathbb{R}$:

Definition. (*i*) (Hausdorff 1914) A subset $O \subseteq \mathbb{R}^n$ is *open [offen]* if for each $x \in O$ there is an open ball $B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$ with $r > 0$, such that $B_r(x) \subseteq O$.
(*ii*) (Cantor 1884) A subset $C \subseteq \mathbb{R}^n$ is *closed [abgeschlossen]* if it has the following property: For each sequence $x_k \in C$ which converges to some $a \in \mathbb{R}^n$ the limit a is an element of C.

Intuitively, a set is closed when it contains all of its boundary, and it is open, when it does contain no boundary point.

Examples. 1. The interval $[a, b] \subseteq \mathbb{R}$ is closed. Indeed, from $a \leq x_k \leq b$ follows $a \leq \lim x_k \leq b$. On the other hand, the interval (a, b) is open. Let $x \in (a, b)$, and $r := \min\{|x - a|, |x - b|\} > 0$. Then $(x - r, x + r) \subseteq (a, b)$ is the required open ball.

2. The interval $(0, 1]$ is not closed: The sequence $(\frac{1}{k})_{k \in \mathbb{N}}$ has the limit 0 which is not contained in $(0, 1]$. On the other hand, the interval is not open, since it does not contain any ball about 1: Indeed, for each $r > 0$ we have $(1 - r, 1 + r) \not\subseteq (0, 1]$.

3. $B_r(y)$, the open ball, is indeed open (H9.3). The set

$$\overline{B}_r(y) := \{x \in \mathbb{R}^n \mid \|x - y\| \leq r\}$$

is closed; it is called the *closed ball.*

4. $\mathbb{R}^n$ and the empty set are open and closed (check!).

Closed and open sets are complimentary:

Theorem 11. *A subset $C \subseteq \mathbb{R}^n$ is closed if and only if its* complement $\mathbb{R}^n \setminus C$ *is open.*

Proof. "⇒", indirectly: Suppose that $\mathbb{R}^n \setminus C$ is not open. Then there is $a \in \mathbb{R}^n \setminus C$ such that each ball $B_r(a)$ contains a point of C. In particular, for each $k \in \mathbb{N}$ we can set $r = \frac{1}{k}$

to obtain such points $x_k \in B_{1/k}(a) \cap C$. Hence $\|x_k - a\| \to 0$ and so x_k converges to $a \notin C$, contradiction.

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"$\Leftarrow$": We consider a sequence $x_k \in C$ which converges to $a \in \mathbb{R}^n$. Suppose $a \notin C$. Since $\mathbb{R}^n \setminus C$ is open there is $r > 0$ with $B_r(a) \cap C = \emptyset$. But then $\|x_k - a\| \geq r$ for all $k \in \mathbb{N}$ and hence $x_k \to a$ cannot hold, so that $a \in C$ must hold. ☐

Perhaps another characterization of closed sets will explain what these sets look like. Given a set $X \subseteq \mathbb{R}^n$, a point $a \in \mathbb{R}^n$ is an *accumulation point* (or *limit point*) *[Häufungspunkt] of* X if there exists a sequence $x_k \in X - \{a\}$ with $x_k \to a$.

Examples. 1. The set of accumulation points for the open ball $B_r(y)$ is the closed ball $\overline{B}_r(x)$.

2. $\mathbb{Z} \subseteq \mathbb{R}$ has no accumulation points.

3. $\{\frac{1}{k} \mid k \in \mathbb{N}\}$ has 0 as the only accumulation point.

By definition, a set X is closed if and only if it contains all its accumulation points.

Let us consider subsets of $\mathbb{R}$. Which kind of subset contains a maximal element? The open set $(0, 1)$ does not contain a maximum. What about closed sets? On the one hand $[0, 1]$ does contain a maximum, but on the other hand $\mathbb{R}$ does not! So let us define:

Definition. A subset $K \subseteq \mathbb{R}^n$ is *compact [kompakt]* if it is closed and bounded.

Example. 1. A finite subset $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^n$ is compact.
2. A closed ball is compact.
3. $\mathbb{R}^n$ is not compact.

For dimension 1, let us prove what the above examples hinted at:

Theorem 12. *A compact subset $K \subseteq \mathbb{R}$ contains its supremum and infimum.*

Proof. Since K is bounded, we have $x \leq C$ for all $x \in K$. Hence $s := \sup K$ exists (Thm. I32). Let $a_k \in K$ be a sequence with $a_k \to s$. Since K is closed, $s = \lim_{k\to\infty} a_k \in K$. Similarly for the infimum. ☐

2.5. **Minima and maxima of continuous functions.** Whenever functions $f\colon X \to \mathbb{R}$ come up, an important issue is to determine maxima and minima. If these exist, then differentiation helps to locate them. Here, however, we ask a more basic question: Does a function f attain its supremum?

Examples. 1. $\frac{1}{x}$ does not have a maximum on $(0,1)$ (the range in unbounded).
2. $\frac{1}{1+x^2}$ does not take a minimum on $\mathbb{R}$.

However, over compact sets we have:

Theorem 13. *Let $K \subseteq \mathbb{R}^n$ be a nonempty compact set.*
(i) [Weierstrass' Hauptlehrsatz, 1861] *If $f\colon K \to \mathbb{R}$ is continuous, then f* attains a maximum and minimum over K, *that is, there are points $a, b \in K$ such that*

$$f(a) \le f(x) \le f(b) \qquad \textit{for all } x \in K.$$

(ii) *If $f\colon K \to \mathbb{R}^m$ is continuous then the range $f(K)$ is compact.*

In case (i), the statement is $f(x) \in \big[f(a), f(b)\big]$. Note that in case that K is a closed bounded interval, the intermediate value theorem gives in fact $f(K) \supseteq \big[f(a), f(b)\big]$.

Proof. (i) Let

$$B := \sup\big\{f(x) \mid x \in K\big\} \in \mathbb{R} \cup \{\infty\},$$

where we formally write $B = \infty$ in case the range of f is unbounded. There is a sequence $x_k \in K$ with $f(x_k) \to B$. In general, (x_k) need not converge (consider the example that f is constant). But the sequence (x_k) is bounded, as K is bounded. Now recall the theorem of Bolzano-Weierstrass in $\mathbb{R}^n$, Corollary I,41: The sequence (x_k) has a a subsequence x_{ν_k} which converges to some point $b \in \mathbb{R}^n$. Since K is also closed, we have $b \in K$. Now we use the continuity of f which gives

$$f(b) = \lim_{k\to\infty} f(x_{\nu_k}) = B.$$

Since $f(b) \in \mathbb{R}$, we can conclude $B \ne \infty$.

The same ideas can be applied to construct the minimum. Here, $A = \inf f(K) \in \mathbb{R} \cup \{-\infty\}$, and the construction of a sequence with limit $a \in K$ is as above. In this case, using continuity we conclude $A = f(a) \ne -\infty$.

(ii) The scalar-valued function $\|f\|\colon K \to \mathbb{R}$ is continuous (since $\|.\|$ is continuous, see Thm. 6) and so it takes a maximum C according to part (i). Therefore $f(K)$ is bounded by C. It remains to show it is closed. If (y_k) is a sequence in $f(K)$ which converges to y, then we can find preimages $x_k \in K$ (not necessarily unique) with $y_k = f(x_k)$. As in part (i) we select a subsequence x_{ν_k} which converges to some point $x \in K$. Consequently,

$$f(x) = f\big(\lim_{k\to\infty} x_{\nu_k}\big) \overset{f \text{ cts.}}{=} \lim_{k\to\infty} f(x_{\nu_k}) = \lim_{k\to\infty} y_{\nu_k} = y,$$

and so indeed the limit y is in $f(K)$.

□

3. FURTHER CHARACTERIZATIONS OF CONTINUITY

3.1. **The ε-δ-test.** For the theory, the following test for continuity will be indispensible (see the discussion of uniform continuity below).

Theorem 14 (ε-δ-test)**.** *Let $X \subseteq \mathbb{R}^n$ and $f\colon X \to \mathbb{R}^m$. Then f is continuous at $a \in X$, if and only if the following holds: For each $\varepsilon > 0$ there is $\delta > 0$, such that*

$$\|f(x) - f(a)\| < \varepsilon \qquad \text{for all } x \in X \text{ with } \|x - a\| < \delta. \tag{7}$$

Thus each ε-ball $B_\varepsilon\big(f(a)\big)$ about $f(a)$ contains the entire image of some δ-ball about a. So all points in $B_\delta(a) \cap X$ meet the error bound ε for $f(a)$.

Note. δ depends on a und ε. In general, $\delta(\varepsilon)$ measures how much f oscillates near a.

Examples. 1. Let $p \in \mathbb{R}^n$ and $f(x) := \|x - p\|$. We expect that $\delta := \varepsilon$ works. Indeed,

$$|f(x) - f(a)| = \big|\|x-p\| - \|a-p\|\big| \overset{\text{sharpened . }\Delta\text{-inequ.}}{\le} \big\|x - p - (a-p)\big\| = \|x - a\|,$$

that is, for $\|x - a\| < \delta$ we have verified $|f(x) - f(a)| < \varepsilon$.

2. Let the sign function $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$ be defined by $\operatorname{sgn}(x) := 1$ for $x > 0$, $\operatorname{sgn}(x) := -1$ for $x < 0$ and $\operatorname{sgn}(0) := 0$. Then sgn is not continuous at $x = 0$. Indeed, for $\varepsilon := \frac{1}{2}$, whatever $\delta > 0$ is, we have $|f(\frac{\delta}{2}) - f(0)| = 1 \not< \varepsilon$.

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3. For $f(x) = x^2$ the graph gets steeper as $|a|$ becomes larger. Hence we expect that $\delta \to 0$ as $a \to \infty$. We claim that $\delta := \min\big(1, \frac{\varepsilon}{2|a|+1}\big)$ works. Indeed, for x with $|x - a| < \delta$:

$$|x^2 - a^2| = |x-a|\,|\underbrace{x+a}_{x-a+2a}| \overset{\Delta\text{-inequ.}}{\le} \underbrace{|x-a|}_{<\delta}\big(\underbrace{|x-a|}_{<\delta\le 1} + |2a|\big) < \frac{\varepsilon}{2|a|+1}(1 + 2|a|) = \varepsilon$$

The last example is not at all straightforward. To understand why this is so, note that the ε-δ-test is a quantitative measure of the continuity of f at a, while the limit test is merely qualitative. Hence the ε-δ test is often harder to check than the limit test.

Proof. "$\Leftarrow$": We assume the ε-δ-condition for $a \in X$. We need to show that for each sequence $x_k \to a$ in the domain, the image sequence satisfies $f(x_k) \to f(a)$. Let $\varepsilon > 0$ be arbitrary, and pick δ by (7). Since $x_k \to a$ we can choose $N \in \mathbb{N}$ such that $\|x_k - a\| < \delta$ for all $k \ge N$. But (7) then implies $\big\|f(x_k) - f(a)\big\| < \varepsilon$ for all $k \ge N$. As ε was arbitrary, this gives $f(x_k) \to f(a)$, as desired.

"$\Rightarrow$". Assume the limit test holds. We show indirectly that for each $\varepsilon > 0$ there is $\delta > 0$ with (7).

Suppose for a particular error bound $\varepsilon > 0$ there were no $\delta > 0$ satisfying the condition (7). In particular, (7) could not be satisfied with any $\delta = \frac{1}{k}$, where $k \in \mathbb{N}$. Thus there exist $x_k \in X$ with $\|x_k - a\| < \frac{1}{k}$, so that $\|f(x_k) - f(a)\| \geq \varepsilon$. Therefore we have $x_k \to a$ but not $f(x_k) \to f(a)$, contradicting the limit test. $\square$

A property as simple as important can easily be derived from the ε-δ-test:

Corollary 15. *Let $X \subseteq \mathbb{R}^n$ and $f\colon X \to \mathbb{R}^m$ be continuous with $f(a) \neq 0$ for $a \in X$. Then there is $\delta > 0$ such that $f(x) \neq 0$ for all $x \in B_\delta(a)$.*

Proof. For $\varepsilon := \|f(a)\|$ let us choose δ according to the ε-δ-condition. Then for all x with $\|x - a\| < \delta$ we verify:

$$\|f(x)\| = \|f(a) + f(x) - f(a)\| \overset{\text{sharp. }\Delta\text{-inequ.}}{\geq} \|f(a)\| - \underbrace{\|f(x) - f(a)\|}_{<\varepsilon = \|f(a)\|} > 0.$$

$\square$

3.2. **The topological characterization in terms of open sets.** Our two tests for continuity literally work for metric spaces: If $f\colon X \to Y$ then we can consider sequences (or balls) with respect to the metrics d_X and d_Y of X and Y, respectively. There is third characterization of continuity, which we will need when discussing the inverse mapping theorem. At the same time, it suggests how to define continuity the most general setup that is, for maps between so-called topological spaces.

The chararcterization is in terms of open sets. Note that the continuous image of an open set is not necessarily open: The interval $(-1, 1)$ maps under the continuous function x^2 to the interval $[0, 1)$, which is not open. But for inverse images we have:

Theorem 16. *Let $X \subseteq \mathbb{R}^n$ be open. A mapping $f\colon X \to \mathbb{R}^m$ is continuous if and only for each open set $O \subseteq \mathbb{R}^m$ the preimage $f^{-1}(O)$ is open again.*

Examples. 1. For the continuous map x^2, the preimage of the open interval $(0, 4)$ is $(-2, 0) \cup (0, 2)$, which is indeed open.

2. For the discontinuous map f with $f(x) := 0$ for $x \leq 0$ and $f(x) := 1$ for $x > 0$ we have that the preimage of $(-\frac{1}{2}, \frac{1}{2})$ is $(-\infty, 0]$, which is not open.

Proof. "$\Leftarrow$": Let $y = f(x)$. We want to assert the ε-δ-condition at x. Let $\varepsilon > 0$. The ball $B_\varepsilon(y)$ is an open subset of $\mathbb{R}^n$. By assumption its preimage is open again. Hence the preimage contains an open ball about each of its points. In particular, it contains some ball $B_\delta(x)$. Hence (7) holds.

"$\Rightarrow$": Let f be continuous and $x \in f^{-1}(O)$. We need to show that some δ-ball about x is entirely contained in the set $f^{-1}(O)$. Since O is open, we can choose $\varepsilon > 0$, so that

$B_\varepsilon\big(f(x)\big) \subseteq O$. Using the ε-δ-test we find $\delta' > 0$ such that $B_{\delta'}(x) \cap X$ maps into $B_\varepsilon\big(f(x)\big)$. It can be that $B_{\delta'}(x)$ is not entirely contained in X; so if necessary, we need to shrink the radius further as follows. Since X is open, and $x \in X$, we have $B_r(x) \subseteq X$. Note that for $\delta := \min(\delta', r)$ the ball $B_\delta(x)$ maps into $B_\varepsilon\big(f(x)\big)$, which is, the set $f^{-1}(O)$ contains the entire ball $B_\delta(x)$. ☐

4. Logarithm and general powers

We consider functions from $\mathbb{R}$ to $\mathbb{R}$, namely the inverse functions of exp and x^k.

4.1. **The logarithm.** By Corollary 10(i), the function $\exp\colon \mathbb{R} \to \mathbb{R}^+$ has an inverse function, $\log := \exp^{-1}\colon \mathbb{R}^+ \to \mathbb{R}$, that is, $y = \exp(x)$ if and only if $x = \log y$. It is called the *(natural) logarithm.* The logarithm is continuous, as follows from the following general statement, which once again applies the $\varepsilon - \delta$-test:

Theorem 17. *Let $f : (a,b) \to \mathbb{R}$ be strictly monotone and continuous. Then the inverse function $f^{-1}\colon f\big((a,b)\big) \to \mathbb{R}$ is strictly monotone and continuous as well.*

Proof. Monotonocity is obvious (check!).

To see f^{-1} is continuous, we use the ε-δ-test. Let $x \in (a,b)$ and $\varepsilon > 0$ be given. We consider $I_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$ and assume that ε is small enough so that $[x - \varepsilon, x + \varepsilon] \subseteq (a,b)$; note that it is sufficient to verify the ε-δ-test for small $\varepsilon > 0$.

Since f is continuous the intermediate value theorem implies that $f\big([x - \varepsilon, x + \varepsilon]\big)$ is an interval; in the strictly increasing case, it must be the interval $\big[f(x-\varepsilon), f(x+\varepsilon)\big]$. Moreover, a strictly monotone function is bijective, and so the inverse function f^{-1} maps the open interval $f\big(I_\varepsilon(x)\big)$ onto the open interval $I_\varepsilon(x)$.

Now we only need to choose $\delta > 0$ such that $I_\delta\big(f(x)\big) := \big(f(x) - \delta, f(x) + \delta\big) \subseteq f\big(I_\varepsilon(x)\big)$; for instance $\delta := \min\big(|f(x + \varepsilon) - x|, |x - f(x - \varepsilon)|\big)$ works. Then $f^{-1}\big(I_\delta(f(x))\big) \subseteq I_\varepsilon(p)$, which shows the inverse function f^{-1} is continuous at $f(x)$. ☐

Problem. Sketch the graph of a continuous and strictly monotone function, defined on the union of two intervals $(a,b] \cup (c,d)$, such that the inverse function is *not* continuous.

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Theorem 18. *The logarithm* $\log\colon \mathbb{R}^+ \to \mathbb{R}$ *is continuous, strictly increasing, and satisfies*

$$\log(xy) = \log x + \log y \qquad \text{for } x, y \in \mathbb{R}^+. \tag{8}$$

Proof. We have

$$\exp\big(\log(xy)\big) = xy = \exp(\log x)\exp(\log y) \overset{(3)}{=} \exp(\log x + \log y).$$

Using the injectivity of exp gives the claim. ☐

4.2. **Powers.** Like for exp we can also combine Corollary 10(*ii*) with Theorem 17 to obtain:

Theorem 19. *For each $k \in \mathbb{N}$, the power $f\colon \mathbb{R}_0^+ \to \mathbb{R}$, $f(x) = x^k$ has a strictly monotone and continuous inverse function, denoted $\sqrt[k]{.}\colon \mathbb{R}_0^+ \to \mathbb{R}$.*

Our goal is to extend the definition of the power function to real and complex exponents. To extend to rational exponents is easy by composition:

Definition. For each $q = \frac{m}{k} \in \mathbb{Q}$ with $m \in \mathbb{Z}$, $k \in \mathbb{N}$ let

$$.^q\colon \mathbb{R}^+ \to \mathbb{R}, \qquad x^q := \sqrt[k]{x^m}.$$

For each $q \in \mathbb{Q}$ the function $x \mapsto x^q$ is continuous according to the composition rule Thm. 7.

We now consider the power function a^q for fixed basis $a \in \mathbb{R}^+$ and variable exponent $q \in \mathbb{Q}$. It is somewhat involved to assert that $q \mapsto a^q$ is continuous. Taking this for granted, we could define a^x as the unique continuous function on $\mathbb{R}$ which agrees with a^x for $x \in \mathbb{Q}$. Instead, we now pursue a less intuitive approach, which is, however, technically simpler and works for complex powers as well.

For each $a > 0$ we define a function

$$f_a\colon \mathbb{C} \to \mathbb{C}, \quad \text{with} \quad f_a(z) := \exp(z \log a).$$

Then, for instance, $f_a(2) = \exp(2\log a) = a^2$; more generally, f_a extends the rational powers:

Theorem 20. *The function f_a is continuous, and for $q \in \mathbb{Q}$ holds*

$$f_a(q) = a^q.$$

Proof. Continuity follows from the continuity of exp and log using the composition rule.

f_a satisfies the same functional equation as exp:

$$f_a(z+w) = \exp\big((z+w)\log a\big) \overset{(3)}{=} \exp\big(z\log a\big)\exp\big(w\log a\big) = f_a(z)f_a(w). \tag{9}$$

Therefore, for $m \in \mathbb{N}$ we have $f_a(m) = f_a(1+\ldots+1) \overset{(9)}{=} \big(f_a(1)\big)^m = \big(\exp(1\log a)\big)^m = a^m$. Using $f_a(0) = 1$ and (9) we also obtain $f_a(-z) = \frac{1}{f_a(z)}$. In conclusion, the result is

$$f_a(m) = a^m \qquad \text{for all } m \in \mathbb{Z}. \tag{10}$$

Write now $q = \frac{m}{k}$ with $m \in \mathbb{Z}$ and $k \in \mathbb{N}$. Using Thm. 19 we can take the k-th root of $\big(f_a(\frac{m}{k})\big)^k = f_a\big(\frac{m}{k} + \ldots + \frac{m}{k}\big) = f_a(m)$ to establish the claim:

$$f_a(q) = f_a\Big(\frac{m}{k}\Big) = \sqrt[k]{f_a(m)} \overset{(10)}{=} \sqrt[k]{a^m} = a^{\frac{m}{k}} = a^q \qquad \square$$

The theorem asserts that the following definition is well-defined, that is, it is unambiguous for the previously defined case of rational powers:

Definition. The *general power* is defined by

$$a^z := \exp(z \log a) \qquad \text{for } z \in \mathbb{C},\ a > 0.$$

Let $x > 0$ and consider $a \searrow 0$. Then $x \log a \to -\infty$, and so

$$\lim_{a \to 0} a^x = \lim_{a \to 0} \exp(x \log a) = 0.$$

Hence for real $x > 0$ we also set $0^x := 0$; this makes $a \mapsto a^x$ into a continuous function on $a \in [0, \infty)$.

Problem. For which $z \in \mathbb{C}$ does $\lim_{a \to 0} a^z$ have a limit?

When we specialize to $a := e = \exp(1)$ we obtain

$$e^z = \exp(z \log e) = \exp(z) \qquad \text{for all } z \in \mathbb{C},$$

and thereby justify the familiar notation for the exponential function. We will use it from now on.

Let us finally list the arithmetic rules for powers, which hold for complex z just as they do for rational values $(a, b > 0)$:

$$a^z a^w \overset{(9)}{=} a^{z+w},$$

$$a^z b^z = e^{z \log a} e^{z \log b} \overset{(3)}{=} e^{z(\log a + \log b)} \overset{(8)}{=} e^{z \log(ab)} = (ab)^z,$$

$$\Big(\frac{1}{a}\Big)^z = e^{z \log(1/a)} = e^{-z \log a} = a^{-z}$$

Problem. Show $(a^x)^w = a^{xw}$. Why do we need to require $x \in \mathbb{R}$, $w \in \mathbb{C}$?

4.3. **Growth rates of** exp **and** log. We write $\lim_{x \to \infty} f(x) = c \in \mathbb{R} \cup \infty \cup \{-\infty\}$, if for each sequence $x_n \in D \subseteq \mathbb{R}$ which diverges to ∞ we have $\lim_{n \to \infty} f(x_n) = c$, and provided there is at least one such sequence. Similarly for $\lim_{x \to -\infty} f(x)$.

Examples. 1. $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to -\infty} e^x = 0$ can be derived from the values of exp on $\mathbb{Z}$ and its monotonicity. (Compare the proof of Corollary 10.)

2. exp increases stronger than any power (H12.5):

$$\lim_{x\to\infty} \frac{e^x}{x^k} = \infty \qquad \text{for each } k \in \mathbb{N} \tag{11}$$

3. log increases weaker than any power:

$$\lim_{x\to\infty} \frac{\log x}{x^a} = 0 \qquad \text{for each } a > 0 \tag{12}$$

Proof:

$$\frac{\log x}{x^a} = \frac{\log x}{e^{a\log x}} = \frac{1}{\frac{e^{a\log x}}{\log x}} = \frac{\frac{1}{a}}{\frac{e^{a\log x}}{a\log x}}$$

Let us now take the limit $x \to \infty$. Then also $\log x \to \infty$, as log is increasing with range $\mathbb{R}$. Therefore, also $y := a\log x \to \infty$ and

$$\lim_{x\to\infty} \frac{\log x}{x^a} = \lim_{y\to\infty} \frac{\frac{1}{a}}{\frac{e^y}{y}} = \frac{1}{a} \lim_{y\to\infty} \frac{1}{\frac{e^y}{y}}.$$

By (11) the denominator diverges to ∞, and so the fraction converges to 0 (check!)

4.4. **Landau-symbols o and O.** We write

$$f(x) = o\big(h(x)\big) \quad \text{as } x \to \infty \qquad :\Longleftrightarrow \qquad \lim_{x\to\infty} \frac{f(x)}{h(x)} = 0.$$

Computer scientists write $f(x) = \omega\big(h(x)\big)$.

Examples. 1. For each $a > 0$ we have $\log x = o(x^a)$ by (12).
2. The limit (11) is equivalent to $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$ and therefore $x^k = o(e^x)$ for all $k \in \mathbb{N}$.

Moreover, we write

$$f(x) = O\big(h(x)\big) \quad \text{as } x \to \infty \qquad :\Longleftrightarrow \qquad \frac{f(x)}{h(x)} \text{ stays bounded as } x \to \infty,$$

i.e., there exist $C, x_0 \in \mathbb{R}$ with $\left|\frac{f(x)}{h(x)}\right| \le C$ for all $x > x_0$.

Examples. 1. $p(x) := a_0 + a_1 x + \ldots + a_n x^n = O(x^n)$.
2. If $f(x) = o\big(h(x)\big)$ then certainly $f(x) = O\big(h(x)\big)$ (check!).

The notation $O(x)$ and $o(x^5)$ does not denote functions, but is only short hand for the limit properties of functions.

For Computer Science, $O(n\log n)$ is important: This is the runtime of an algorithm which sorts a list with n entries (to alphabetical order, according to size, etc.). Note that $17n\log n + 1000n = O(n\log n)$, that is, for *fixed* n the Landau-notation does not imply any information on the actual size of a quantity. Only the asymptotic growth order as $n \to \infty$ is prescribed.

More generally, Landau symbols are in use for any limit $x \to a$:

Example. $e^x = 1 + O(x)$ as $x \to 0$ by (6).

`26. Lecture, Thursday, 6. Feb. 03` ———

5. Trigonometric functions

5.1. **Sine and Cosine.** We start from scratch and introduce the trigonometric functions by their series representations. Properties like periodicity are not at all obvious from this definition, but will be established in the course of the section. Only in the next chapter we will use integration to derive the geometric properties which make the functions *trigonometric*, that is triangle-measuring.

Definition. For $z \in \mathbb{C}$ we define the series *sine [Sinus]* and *cosine [Cosinus]* by

$$\cos z := 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \mp \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}, \tag{13}$$

$$\sin z := z - \frac{z^3}{3!} + \frac{z^5}{5!} \mp \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}. \tag{14}$$

Note that $\cos x, \sin x \in \mathbb{R}$ for $x \in \mathbb{R}$.

The series sin and cos contain any other term of the exponential series, up to sign. Thus they are closely linked to exp:

Theorem 21. (i) *The series* $\cos z$ *and* $\sin z$ *converge absolutely for all* $z \in \mathbb{C}$.
(ii) $e^{iz} = \cos z + i \sin z$ (Euler formula).
(iii)

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

(iv) sin *and* cos *are continuous functions on* $\mathbb{C}$.

Proof. (i) This can be verified using the ratio test, or using majorization with the series $\exp(|z|)$ (see H11.4).

(ii) Since $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc., we obtain

$$\exp(iz) = 1 + iz + -\frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} + \ldots .$$

This establishes a version of the Euler formula for the (appropriate) partial sums of the three series. By convergence of all three series, the Euler formula follows.

(iii) From considering partial sums we can infer $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$ for all $z \in \mathbb{C}$. (That is, cos is an *even function*, and sin is *odd.*) Thus the Euler formula gives $e^{-iz} = \cos z - i \sin z$. Adding or subtracting the Euler formula from this gives the result.

(*iv*) Using that exp is continuous, we find that the right hand sides of (*iii*) are continuous. □

Remark. We want to give a geometric picture for the Euler formula. Namely, we want to reason for the fact that when x is real, $e^{it} = \cos t + i \sin t$ travels on the unit circle $\mathbb{S}^1 = \big\{|z| = 1\big\} \subseteq \mathbb{C}$. Indeed, using the fact that $\exp(\overline{z}) = \overline{\exp(z)}$ we find $|e^{it}|^2 = 1$, see H12.4. Now, given a rectangular triangle with hypothenuse of length 1 and angle t (measured in radians) why are $\sin t$ and $\cos t$ the edgelengths of the cathetes? Were we to use complex differentiation, we could compute the velocity vector $(e^{it})' = ie^{it}$. It has modulus $|ie^{it}| = |i||e^{it}| = 1$; that is, e^{it} has unit speed on the unit circle. Upon integration this gives: The arc-length on the unit circle (from $0 = e^0$ to e^{it}) must be t. Thus the point e^{it} encloses the angle t with the real axis, measured in radians.

Let us now derive the well-known addition formulas for sine and cosine from evaluating $e^{i(x+y)}$ in two different ways. On the one hand, using the functional equation of exp and the Euler formula we have

$$\begin{aligned} e^{i(x+y)} &= e^{ix}e^{iy} = (\cos x + i \sin x)(\cos y + i \sin y) \\ &= \cos x \cos y - \sin x \sin y + i\big(\cos x \sin y + \sin x \cos y\big); \end{aligned} \tag{15}$$

on the other hand

$$e^{i(x+y)} = \cos(x+y) + i \sin(x+y). \tag{16}$$

Note that sine and cosine of real numbers are real once again. So the right hand sides of (15) and (16) are written in the form 'real plus imaginary part'. Equating these parts, we obtain the *addition theorems*

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad \text{and} \quad \sin(x+y) = \cos x \sin y + \sin x \cos y. \tag{17}$$

In particular, we obtain:
1. Setting $y = -x$ in the cosine formula gives $1 = \cos^2 x + \sin^2 x$.
2. Setting $x = y$ gives $\cos(2x) = \cos^2 x - \sin^2 x$ and $\sin(2x) = 2 \cos x \sin x$.

Exercise: Interpret $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$ in terms of polar coordinates: What is the polar angle, what the modulus of e^z? Given this interpretation, explain why the functional equation is a consequence.

5.2. **The number π.** We take a route as quick as ungeometric, and will define the circle number π as two times the first zero of $\cos\colon \mathbb{R}^+ \to \mathbb{R}$. At this point, we do not yet know that cos and sin have any zeros (apart from $\sin 0 = \operatorname{Im}(e^0) = 0$). The following will help us to locate the desired zeros.

Lemma 22. *For $x \in (0,2]$ we have the inclusions*

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}, \tag{18}$$

$$0 < x - \frac{x^3}{6} < \sin x < x. \tag{19}$$

Proof. The series $\sin x$ and $\cos x$ alternate. Recall the Leibniz test Thm. I,45 and its proof: If $a_n \geq 0$ is a decreasing null sequence then $s := a_0 - a_1 + a_2 - a_3 \pm \dots$ exists and

$$a_0 - a_1 \pm \dots - a_{2k+1} \leq s \leq a_0 - a_1 \pm \dots + a_{2k} \qquad \text{for each } k \geq 0. \tag{20}$$

The series sine (14) is the alternating sum of the terms $a_k := \frac{|x^{2k+1}|}{(2k+1)!}$. For $x \in (0,2]$ they are decreasing as

$$\frac{x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{x^{2k+1}} = \frac{x^2}{(2k+3)(2k+2)} < 1 \qquad \text{for all } k \geq 0.$$

We apply (20) with $k = 0$ to find that the first two partial sums estimate the series as claimed.

For cosine (13) we have $a_k := \frac{x^{2k}}{(2k)!}$. Provided that $0 < x \leq 2$ these terms decrease, starting at the second term:

$$\frac{x^{2k+2}}{(2k+2)!} \cdot \frac{(2k)!}{x^{2k}} = \frac{x^2}{(2k+1)(2k+2)} < 1 \qquad \text{for } k \geq 1.$$

Estimate (18) follows. ☐

`27. Lecture, Thursday, 13. Feb. 03` ---

Theorem 23. *Cosine has exactly one zero in the interval $[0,2]$.*

Proof. By (18) we have $\cos 0 = 1 > 0$ and $\cos 2 < 1 - \frac{4}{2} + \frac{16}{24} = 1 - 2 + \frac{2}{3} = -\frac{1}{3}$. Now cos is a continuous function (Thm. 21(*iv*)), and so the Intermediate Value Theorem 9 establishes the existence of a zero.

It remains to show uniqueness. We show cos is decreasing on $[0,2]$. For $x, y \in \mathbb{R}$ we set $u := \frac{y+x}{2}$, $v := \frac{y-x}{2}$. Then

$$\begin{aligned} \cos x - \cos y &= \cos(u - v) - \cos(u + v) \\ &\overset{(17)}{=} \cos u \underbrace{\cos(-v)}_{\cos v} - \sin u \underbrace{\sin(-v)}_{-\sin v} - \cos u \cos v + \sin u \sin v \\ &= 2 \sin u \sin v = 2 \sin \frac{y+x}{2} \sin \frac{y-x}{2}. \end{aligned} \tag{21}$$

Let us now specialize to $0 < x < y \leq 2$. Then $0 < \frac{y \pm x}{2} \leq 2$ and so (19) gives $\sin \frac{y \pm x}{2} > 0$. This proves $\cos x > \cos y$, as desired. ☐

Definition. We denote two times the zero of cos on $[0,2]$ by π.

Remarks. 1. From a computational point of view, this definition is certainly impractical. Efficient ways to compute π are discussed in Section 8.11 (S. 132/33) of [K], where further references are also provided.

2. Using integration, we will later verify the classical definitions of π, namely that the unit circle has area π, and circumference 2π.

5.3. **Periodicity of sine, cosine,** exp. Let us now draw consequences of the identity $\cos\frac{\pi}{2} = 0$.

(*i*) We claim that $\sin\frac{\pi}{2} = 1$. Indeed, sin is positive on $(0,2]$ and $1 = \cos^2\frac{\pi}{2} + \sin^2\frac{\pi}{2} = \sin^2\frac{\pi}{2}$.

(*ii*) Therefore, using the functional equation:

$$e^{ik\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^k = \Big(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\Big)^k \overset{(i)}{=} i^k.$$

In particular,

$$e^{\pi i} = -1;$$

this formula relates four famous numbers, and, arguably, can be considered the most beautiful formula of mathematics. Squaring gives $e^{2\pi i} = 1$.

Corollary 24. *We have the following periodicities, for all* $z \in \mathbb{C}$*:*

$$\begin{aligned} e^{z+2\pi i} &= e^z, & \cos(z+2\pi) &= \cos z, & \sin(z+2\pi) &= \sin z \\ e^{z+\pi i} &= -e^z, & \cos(z+\pi) &= -\cos z, & \sin(z+\pi) &= -\sin z \end{aligned}$$
$$\cos\big(\frac{\pi}{2} - z\big) = \sin z$$

The last formula follows from the addition theorem:

$$\cos\big(\frac{\pi}{2} - z\big) = \cos\frac{\pi}{2}\cos z + \sin\frac{\pi}{2}\sin z = \sin z.$$

By the corollary, it is enough to know cos on $[0,\frac{\pi}{2}]$, in order to know all values of sin and cos on the reals. In particular, from Theorem 23 we find the following real zeros:

$$\big\{x \in \mathbb{R} \mid \cos x = 0\big\} = \big\{k\pi + \frac{\pi}{2} \mid k \in \mathbb{Z}\big\}, \qquad \big\{x \in \mathbb{R} \mid \sin x = 0\big\} = \big\{k\pi \mid k \in \mathbb{Z}\big\} = \pi\mathbb{Z}$$

The previous facts give us information on the injectivity of $\exp\colon \mathbb{C} \to \mathbb{C}$, which we have not yet disussed. Let us first find the solutions of $e^{x+iy} = 1$. Looking at modulus and imaginary part of

$$1 = e^{x+iy} = e^x\big(\cos y + i\sin y\big)$$

we find that necessarily $|e^x| = 1 \iff x = 0$, and $y \in \pi\mathbb{Z}$. But for y an odd multiple of π we have $\sin y = -1$ and so $e^{iy} = -1$. Hence

$$\big\{z \in \mathbb{C} \mid \exp z = 1\big\} = \big\{x + iy \in \mathbb{C} \mid x = 0,\ \frac{y}{2\pi} \in \mathbb{Z}\big\} = 2\pi i\mathbb{Z}. \tag{22}$$

This allows us to discuss the injectivity of exp. We find

$$\exp(z + w) = \exp(z) \qquad \iff \qquad w \in 2\pi i\mathbb{Z}.$$

Consequently, for each $a \in \mathbb{R}$ the function exp is injective on the strip $\{z \in \mathbb{C} \mid a < \operatorname{Im} z \le a + 2\pi\}$. On such subsets, an inverse function, called the complex logarithm, can be defined. In the class Complex Analysis *[Funktionentheorie]* in the fourth term this will be studied.

As an application, let us finally discuss the roots of unity, that is the solutions of $z^k = 1$ for $k \in \mathbb{N}$. The k complex numbers on the unit circle

$$\zeta_0 := 1, \quad \zeta_1 := \exp\Big(\frac{2\pi i}{k}1\Big), \quad \zeta_2 := \exp\Big(\frac{2\pi i}{k}2\Big), \quad \dots, \quad \zeta_{k-1} := \exp\Big(\frac{2\pi i}{k}(k-1)\Big),$$

obviously fullfil $z^k = 1$. The ζ_j are pairwise different (look at the quotient of a pair and use (22)) and they form the only roots of the equation (again by (22)).

THINGS TO REMEMBER FROM ANALYSIS I

In Analysis II, we will use definitions and arguments from Analysis I without any further reference. Please make sure you know them by heart – then you will be able to follow Analysis II easily.

Definitions:

equivalence relations
group and field axioms
$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}$
- subsets of $\mathbb{R}$: boundedness, sup and inf
- for $\mathbb{R}^n$: scalar product and norm (length); open, closed, compact subsets

sequences:
- convergence and Cauchy, subsequences
- absolut convergence, boundedness, monotonicity

series
continuity definitions: • limit, • ε-δ, • open sets
special functions: exponential series, logarithm, general power, sin, cos
Landau symbols
π

It is not enough to be able to recall the definitions. Rather, try to understand them thoroughly by doing the following. For defined properties (like convergence, continuity, etc.) make sure you know one example, and one counterexample. For definitions like the exponential series it is worthwhile to understand why the definition is significant and, perhaps, why it works.

Statements and theorems:

induction
triangle inequalities
characterizations of the completeness of $\mathbb{R}$:
- Cauchy sequences
- nested intervals
- least upper bounds

Theorem of Bolzano Weierstrass
series:
- $1 + \frac{1}{2^m} + \frac{1}{3^m} + \dots$ for $m \in \mathbb{N}$: the harmonic series, and majorization
- alternating harmonic series, and Leibniz test
- geometric series, majorization, and the ratio test
- decimal expansions

continuous functions:
- Intermediate value theorem,
- $f(a) \neq 0 \Rightarrow f(x) \neq 0$ in a sufficiently small ball about a
- the Weierstrass theorem on the maximum

exp: convergence, functional equation

As for definitions, a good way of understanding results comes from knowing a typical example for a theorem; also, it helps to know an example why a theorem fails when a particular assumption is removed.