

“The low separation axioms (T_0) and (T_1) ”

Thursday 20-12-2001 at 14:29:45 (cet)

A binary relation \preceq on a set is a *quasiorder* if it is *transitive* and *reflexive*. It is called **trivial** if $(\forall x, y)(x \preceq y)$, and is defined to be **discrete** if it agrees with equality; it is said to be a *order* (sometimes simply called a *partial order*) if it is *antisymmetric*, that is, $(\forall x, y)((x \preceq y \text{ and } y \preceq x) \Leftrightarrow (x = y))$. A subset Y of a quasiordered set X is a **lower set** if $(\forall x, y)((x \preceq y \text{ and } y \in Y) \Rightarrow x \in Y)$; the definition of an **upper set** is analogous. On a *topological space* X set $x \preceq y$ iff every *neighborhood* of x is a neighborhood of y . This definition introduces a quasiorder \leq , called the **specialisation quasiorder** of X . This terminology originates from algebraic geometry, s. [17, II, p. 23]. The specialisation quasiorder is trivial iff X has the *trivial topology*. The *closure* \bar{Y} of a subset Y of a topological space X is precisely the lower set $\downarrow Y$ of all $x \in X$ with $x \preceq y$ for some $y \in Y$. The singleton closure $\overline{\{x\}}$ is the lower set $\downarrow\{x\}$, succinctly written $\downarrow x$. The intersection of all neighborhoods of x is the upper set $\uparrow x = \{y \in X : x \preceq y\}$.

A topological space X is said to satisfy the **separation axiom** (T_0) (or to be a T_0 -**space**), and its topology \mathcal{O}_X is called a T_0 -**topology**, if the specialisation quasiorder is an order; in this case it is called the **specialisation order**. The space X is said to satisfy the **separation axiom** (T_1) (or to be a T_1 -**space**), and its topology \mathcal{O}_X is called a T_1 -**topology**, if the specialisation quasiorder is discrete. The terminology for the hierarchy (T_n) of separation axioms appears to have entered the literature 1935 through the influential book by Alexandroff and Hopf [3] in a section of the book called „Trennungsaxiome“ (pp. 58 ff.). A space is a T_0 -space iff

- (0) for two different points there is an open set containing precisely one of the two points,

and it is a T_1 -space iff

- (1) every singleton subset is closed.

Alexandroff and Hopf call postulate (0) „das nullte Kolmogoroffsche Trennungsaxiom“ and postulate (1) „das erste Frechetsche Trennungsaxiom“ ([3, p. 58, 59]), and they attach with the higher separation axioms the names of Hausdorff, Vietoris, and Tietze. In Bourbaki [4], T_0 -spaces are relegated to the exercises and are called «espaces de Kolmogoroff» (s. §1, Ex. 2, p. 89). Alexandroff and Hopf appear to have had access to an unpublished manuscript by Kolmogoroff which appears to have dealt with quotient spaces ([3, p. 61 and p. 619]) which is likely to have been the origin of this terminology to which Alexandroff continues to refer in later papers (see e.g., [2]). Fréchet calls T_1 -spaces «espaces accessibles» ([7, p. 185]). From an axiomatic viewpoint, the postulate (1) is a natural separation axiom for those who base topology on the concept of a closure operator (Kuratowski 1933); Hausdorff’s axiom, called (T_2) by Alexandroff and Hopf, is a natural one if the primitive concept is that of neighborhood systems

(Hausdorff [9] 1914). In [14] Kuratowski joins the terminology by referring to «espaces T_1 » (loc. cit. p. 38).

While today we call a topological space *discrete* if every subset is open, we now call a space **Alexandroff-discrete** if the intersection of open sets is open, or, which amounts to the same, that every upper set with respect to the specialisation quasiorder is open; in [3] this is applied to the ordered set of cells of a simplicial complex, the order being containment. (Alexandroff himself called these spaces „diskret“, s. [2].) On any ordered set the set of all upper sets is an Alexandroff discrete T_0 -topology. In [2] Alexandroff associates with each Alexandroff discrete space a complex; this remains a viable application of T_0 -spaces even for finite topological spaces. While trivially (T_1) implies (T_0) , the Alexandroff discrete topology on any nondiscretely ordered set is a T_0 -topology that is not a T_1 -topology. Any set supports the so-called *cofinite topology* containing the set itself and all complements of finite sets; the cofinite topology is always a *compact* T_1 -topology (failing to be a *Hausdorff* topology whenever the underlying set is infinite).

On any topological space X with topology \mathcal{O}_X , the binary relation defined by $x \equiv y$ iff $x \preceq y$ and $y \preceq x$ (with respect to the specialisation quasiorder \preceq) is an equivalence relation. The *quotient space* X/\equiv endowed with its *quotient topology* $\mathcal{O}_{X/\equiv}$ is a T_0 -space, and if $q_X: X \rightarrow X/\equiv$ denotes the *quotient map* which assigns to each point its equivalence class, then the function $U \mapsto q_X^{-1}(U): \mathcal{O}_{X/\equiv} \rightarrow \mathcal{O}_X$ is a bijection. Moreover, if $f: X \rightarrow Y$ is any continuous function into a T_0 -space, then there is a unique continuous function $f': X/\equiv \rightarrow Y$ such that $f = f' \circ q_X$. As a consequence of these remarks, for most purposes it is no restriction of generality to assume that a topological space under consideration satisfies (T_0) . Thus the *category* of T_0 -spaces is reflective in the category of topological spaces (cf. [1, p. 43]). The topology \mathcal{O}_X of a topological space is a *complete lattice* satisfying the (infinite) distributive law $x \wedge \bigvee_{j \in J} x_j = \bigvee_{j \in J} (x \wedge x_j)$. After Dowker [6] such a lattice is called a **frame**; other authors speak of a **Brouwerian lattice** or a **complete Heyting algebra** (cf., e.g., [13, 8]). The **spectrum** $\text{Spec } L$ of a frame L is the set of all **prime elements** of L [8], and its topology is the so-called **hull-kernel topology** $\mathcal{O}(\text{Spec } L)$. For a T_0 -space X with topology \mathcal{O}_X , the function $x \mapsto X \setminus \overline{\{x\}}: X \rightarrow \text{Spec}(\mathcal{O}_X)$ is a well-defined open embedding. The prime elements of \mathcal{O}_X are of the form $X \setminus A$ where A is a closed set on which the *induced topology* is a *filter basis*; such sets are called **irreducible closed subsets**. Every singleton closure is one of these; the space X is called **sober** if the singleton closures are the only closed irreducible subsets. It is exactly for these T_0 -spaces that the map $X \rightarrow \text{Spec}(\mathcal{O}_X)$ is a homeomorphism; the topology of a sober space determines the space. An infinite set X given the cofinite topology is itself an irreducible closed subset which is not a singleton closure; thus T_1 -spaces need not be sober. The *Hausdorff separation axiom* (T_2) quickly implies that every closed irreducible subset is singleton; that is, Hausdorff spaces are sober. The position of the property (SOB) of being sober in the separation hierarchy thus is as follows

$$(T_0) \Leftarrow (\text{SOB}) \Leftarrow (T_2) \Leftarrow \dots,$$

where none of these implications can be reversed. But (SOB) behaves more like a completeness than as a separation property. Indeed, if X is a T_0 -space then the set X^s of all closed irreducible subsets of X carries a unique topology $\mathcal{O}(X^s)$ making the resulting space homeomorphic to $\text{Spec}(\overline{\mathcal{O}_X})$. The space X^s is sober and the function $s_X: X \rightarrow X^s$, defined by $s_X(x) = \overline{\{x\}}$, is a continuous embedding with dense image such that the function $U \mapsto \sigma_X^{-1}(U): \mathcal{O}(X^s) \rightarrow \mathcal{O}_X$ is a bijection. Moreover, if $f: X \rightarrow Y$ is any continuous function into a sober space, then there is a unique continuous function $f': X^s \rightarrow Y$ such that $f = f' \circ s_X$. The category of sober spaces is reflective in the category of T_0 -spaces. According to these remarks, for many purposes it is no restriction of generality to assume that a T_0 -space and indeed any topological space under consideration is a sober T_0 -space. The space X^s is called the **sobrification** of X .

If L is a frame, the natural frame map $x \mapsto \text{Spec}(L) \setminus \uparrow x: L \rightarrow \mathcal{O}(\text{Spec}(L))$ is surjective; but it is an isomorphism iff and only if for two different elements in L there is a prime element which is above exactly one of the two elements. A complete *Boolean algebra* without atoms (such as the set of (equivalence classes modulo null sets of) Lebesgue measurable subsets of the unit interval) provides an example of a frame without prime elements. The class of frames is the backdrop for a theory of T_0 -spaces “without points” (cf. [13, 16]).

In a *topological group* G , the axiom (T_0) implies all separation axioms up to $(T_{3\frac{1}{2}})$, that is, Hausdorff separation plus *complete regularity*. In fact, the singleton closure $\overline{\{1\}}$ is a closed characteristic and thus normal subgroup N ; and the factor group $G/N = G/\equiv$ is a Hausdorff topological group with the universal property that every morphism from G into a Hausdorff topological group factors through G/N . Every open set U of G satisfies $NUN = U$ and thus is a union of N -cosets. In topological group theory, therefore, the restriction to Hausdorff topological groups is no loss of generality most of the time.

There is a serious watershed between the lower separation axioms (T_0) and (T_1) and the higher separation axioms (T_n) with $n \geq 2$. All of classical general and algebra topology, classical topological algebra and functional analysis is based on Hausdorff separation (T_2) , and Bourbaki [4] shows little interest in anything but Hausdorff spaces, relenting a bit in the second edition. On the other side, in the early history of topology, the theory of T_0 -spaces played a comparatively subordinate role relating mostly to axiomatic matters, but later it developed a momentum of its own as a link between topology, order theory, combinatorics, finite topological spaces, logic, and theoretical computer science. Its significance in algebra and functional analysis is evident in algebraic geometry and the spectral theory of rings, lattices and operator algebras.

Algebraic Geometry. If R is a commutative ring with identity, then the space $\text{spec}(R)$ of prime ideals in the hull-kernel or **Zariski topology** is called the spectrum of R ; the spectrum of a commutative ring is a compact sober T_0 -space, and by a theorem of Hochster every compact sober T_0 -space is so obtained, [10]. In the sense of the spectrum of a frame L mentioned earlier, $\text{spec}(R) = \text{Spec}(L)$ for the lattice L of radical ideals (i.e. those which are intersections of prime ideals, see, e.g., [17, p. 147]). The spectrum is not readily functorial,

but Hofmann and Watkins exhibited a suitable category allowing it to become functorial [12]. The subspace $\max(R) \subseteq \text{spec}(R)$ of maximal ideals is dense; it is a T_1 -space in the Zariski topology, and indeed $\text{Spec}(R)$ is a T_1 -space iff every prime ideal is maximal. The rings that are relevant in algebraic geometry are the polynomial rings $k[X_1, \dots, X_n]$ in n variables over a field k of their homomorphic images. A singleton set in an algebraic variety over an algebraically closed field is an irreducible algebraic variety; hence it is closed. Accordingly, such algebraic varieties are compact T_1 -spaces in their Zariski topology. In particular, the Zariski topology on an algebraic group G over an algebraically closed field is a compact T_1 -topology. (One should note that G is not a topological group with respect to this topology, since the algebraic variety $G \times G$ does not carry the product topology).

Operator Theory. The spectrum of a C^* -algebra A (see, e.g., [5, p. 58ff.]) is the space of all primitive two-sided ideals (i.e., those which are kernels of irreducible representations) in the hull-kernel topology. It is a T_0 -space which in general fails to be a T_1 -, let alone a Hausdorff space; but it is a *Baire space*. If the algebra is *separable*, then the primitive spectrum is a sober space. It is not known whether this is the case in general.

Directed completeness. An ordered set is **directed complete** if every directed subset D has a least upper bound $\sup D$. In the semantics of programming languages such an ordered set is called a **DCPO** (“directed complete partially ordered set”). On a DCPO L , two new structures emerge: (i) an additional transitive relation \ll defined by $x \ll y$ (read “ x is **way below** y ”) iff for all directed subsets D

$$\text{the relation } \sup D \in \uparrow y \text{ implies } D \cap \uparrow x \neq \emptyset,$$

and (ii) a T_0 -topology $\sigma(L)$, where a subset $U \subseteq L$ is in $\sigma(U)$ iff for each directed subset D

$$\text{the relation } \sup D \in U \text{ implies } D \cap U \neq \emptyset.$$

The topology $\sigma(L)$ is called the **Scott topology** of L . A function between DCPOs is continuous with respect to the Scott topologies iff it preserves directed sups. The theory of T_0 -spaces arising as DCPOs with their Scott topologies is particularly rich for those DCPOs in which the set $\downarrow x = \{u \in L : u \ll x\}$ is directed and satisfied $\sup \downarrow x = x$. Such a DCPO is called a **domain** (cf. [8, new edition, Definition I-1.6]). The Scott topology of a domain is sober. For the details see [8]. If a domain L is a lattice, then it is called a **continuous lattice**; this terminology is due to D. S. Scott [15].

Injective T_0 -spaces. A T_0 -space L is called **injective** according to D. S. Scott [15], if for every subspace Y of a T_0 -space X , any continuous function $f: Y \rightarrow L$ extends to a continuous function $F: X \rightarrow L$. Scott proved in 1972 that a T_0 -space is injective iff it is a continuous lattice L endowed with its Scott topology $\sigma(L)$ (cf. [8, II-3]). The theory of continuous lattices has both a (T_0) aspect and a (T_2) aspect: An ordered set is a continuous lattice if and only if it supports a compact Hausdorff topology with respect to which it is a **Lawson semilattice**, that is, a topological semilattice with identity in which every element has a basis

of multiplicatively closed neighborhoods. The topology \mathcal{O}_X of a sober space X is a continuous lattice iff X is locally compact. The order theoretical theory of locally compact sober spaces relies thoroughly on the theory of domains. The category of domains and Scott continuous functions is *cartesian closed*, and Scott produced a canonical construction of continuous lattices L which have a natural homeomorphism onto the space $[L \rightarrow L]$ of continuous self maps; in such a T_0 -space as “universe”, every “element” is at the same time a “function” whence expressions, like $f(f)$, are perfectly consistent. In this fashion Scott produced the first models of untyped λ -calculus of Church and Curry in the form of certain compact T_0 -spaces. Hofmann and Mislove proved that it is impossible to base a model of the untyped λ -calculus on a compact Hausdorff space [11]. For the many beautiful properties of continuous lattices and domains and their relation to general topology in the area of T_0 -spaces see [8].

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