1 Uncountability of $\mathbb{R}$ fails in $\mathsf{Sh}(\mathbb{R})$

as shown by Rosolini and Spitters. Of course, like in a any topos there cannot exist a surjection from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$ but what fails is the statement

$$\forall x \in \mathbb{R}^{\mathbb{N}}.\exists y \in \mathbb{R}.\forall n \in \mathbb{N}.\exists m \in \mathbb{N}.\, |x_n - y| \geq \frac{1}{m}$$

for Dedekind reals $\mathbb{R}$.

Now if this would hold in $\mathsf{Sh}(\mathbb{R})$ by Kripke-Joyal for every open U and $x \in \mathbb{R}^{\mathbb{N}}(U)$, i.e. any continuous function $x : U \times \mathbb{N} \to \mathbb{R}$, there exists a cover $(V_i)_{i \in I}$ of U and continuous $y_i : V_i \to \mathbb{R}$ such that $V_i \Vdash \forall n \in \mathbb{N}.\exists m \in \mathbb{N}.\, |x_n - y_i| \geq \frac{1}{m}$. But then for every $n \in \mathbb{N}$ there exists a cover $(W_{i,j})_{j \in J_i}$ of V_i and $m_{i,n,j} \in \mathbb{N}\backslash\{0\}$ such that

$$|x_n - y_i| \geq \frac{1}{m_{i,n,j}}$$

on $V_{i,j}$.

But for $U = \mathbb{R}$ and x a sequence of functions containing all $f_q(x) = q + x$ and $g_q(x) = q - x$ for $q \in \mathbb{Q}$ this is impossible for the following reason. For all nonempty open $V \subseteq \mathbb{R}$ and continuous maps $y : V \to \mathbb{R}$ there is an x_n such that x_n and y intersect, i.e. have the same value for some argument in V.

The latter claim can be seen as follows. Suppose $f_q(t) \neq y(t)$ and $g_q(t) \neq y(t)$ for all $t \in V$ and $q \in \mathbb{Q}$, i.e. for all $t \in V$ neither $y(t) - t$ nor $y(t) + t$ are elements of $\mathbb{Q}$. Since by the intermediate value theorem every non-constant continuous map from V to $\mathbb{R}$ attains a rational value there exist $a, b \in \mathbb{R}$ with $y(t) - t = a$ and $y(t) + t = b$ for all $t \in V$. But then $2t = b - a$ for all $t \in V$ which is impossible since V being open and nonempty contains infinitely many elements.