A Classical Realizability Model arising from a Stable Model of Untyped Lambda Calculus

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Abstract

In [SR98] it has been shown that λ-calculus with control can be interpreted in any domain D which is isomorphic to the domain of functions from D^ω to the 2-element (Sierpiński) lattice Σ. By a theorem of A. Pitts there exists a unique subset P of D such that $f \in P$ iff $f(\vec{d}) = \bot$ for all $\vec{d} \in P^\omega$. The domain D gives rise to a *realizability structure* in the sense of [Kri11] where the set of proof-like terms is given by P.

When working in Scott domains the ensuing realizability model coincides with the ground model **Set** but when taking D within coherence spaces we obtain a classical realizability model of set theory different from any forcing model. We will show that this model validates countable and dependent choice since an appropriate form of bar recursion is available in stable domains.

1 Introduction

In the first decade of this millenium J.-L. Krivine has developed his theory of *classical realizability*, see e.g. [Kri09, Kri11], for higher order logic and set theory. Whereas intuitionistic realizability is based on the notion of a *partial combinatory algebra* (pca) classical realizability is based on a notion of *realizability algebra* as defined in [Kri11]. Both notions are incomparable since not every pca can be extended to a realizability algebra and there are realizability algebras which do not contain a pca as a substructure. Accordingly, not all classical realizability models appear as booleanizations of intuitionistic realizability models as studied in [vO08].

In the current paper, however, we concentrate on a particular classical realizability model which appears as a boolean subtopos of a *relative realizability* topos (see [vO08]). The starting point for this model is the observation from [SR98] that the recursive domain $D \cong \Sigma^{D^\omega}$ gives rise to a model for λ-calculus with control. (Here $\Sigma = \{\bot, \top\}$ is the 2-element Sierpiński lattice and D^ω is the countable product of D.) Since D is a model of untyped λ-calculus it is in particular a pca. By a theorem of A. Pitts [Pit96] there exists a unique subset P of D such that $t \in P$ iff $t(\vec{s}) = \bot$ for all $\vec{s} \in P^\omega$. Obviously, this subset P forms a sub-pca of D thus giving rise to the relative realizability topos $\mathcal{E} = \mathbf{RT}(D, P)$ as described in [vO08]. Notice that $\top_D \in D \setminus P$ and thus $U = \{\top_D\}$ gives rise to a nontrivial truth value in $\mathcal{E}$ different from both $\top_\mathcal{E}$ and $\bot_\mathcal{E}$. This U (like any subterminal object of $\mathcal{E}$) induces a closure operator (*aka* Lawvere-Tierney topology) $j_U(p) = (p \to U) \to U$ on $\mathcal{E}$. As is well known the subtopos $\mathcal{E}_U$ of j_U-sheaves of $\mathcal{E}$ is boolean.

We will show that $\mathcal{E}_U$ is equivalent to the classical realizability topos $\mathcal{K}$ induced by the realizability structure whose set Λ of terms is D, whose set of stacks Π is D^ω and whose set PL of proof-like terms is P. We will show that $\mathcal{K}$ is equivalent to **Set** when D is the bifree solution of the domain equation $D \cong \Sigma^{D^\omega}$ in Scott domains. However, when considering the solution of $D \cong \Sigma^{D^\omega}$ in the category **Coh** of coherence spaces and Scott continuous and stable maps then the ensuing boolean topos $\mathcal{K}$ is not a Grothendieck topos and thus *a fortiori* not a forcing model. We will show that $\mathcal{K}$ validates all true sentences of first order arithmetic and the principles of countable and dependent choice.

2 Realizability structures induced by $D \cong \Sigma^{D^\omega}$

Quite generally we might consider objects $D \cong \Sigma^{D^\omega}$ in well pointed cartesian closed categories $\mathcal{C}$ with countable products and an object Σ having precisely two global elements (i.e. morphisms $1 \to \Sigma$) $\top$ and $\bot$. The set of global elements of D (which we also denote by D) can be endowed with the structure of a pca as follows: for $t, s \in D$ we define $ts \in D$ as $(ts)(\vec{r}) = t(s.\vec{r})$. For the set Λ of terms we take D and for the set Π of stacks we take D^ω. The push operation sends $t \in \Lambda$ and $\vec{s} \in \Pi$ to $t.\vec{s}$, the stream with head t and tail $\vec{s}$. For every $\vec{s} \in \Pi$ let $\mathsf{k}_{\vec{s}} \in \Lambda$ be defined as $\mathsf{k}_{\vec{s}}(t.\vec{r}) = t(\vec{s})$. The control operator cc is given by $\mathsf{cc}(t.\vec{s}) = t(\mathsf{k}_{\vec{s}}.\vec{s})$. A natural choice for the pole $\perp\!\!\!\perp$ is $\{\langle t, \vec{s}\rangle \mid t(\vec{s}) = \top\}$.

But on this level of generality we do not know how to choose a set PL of "proof-like terms". However, in case D is the bifree solution of $D \cong \Sigma^{D^\omega}$ in some category of domains like

1) cpo's with bottom and Scott continuous functions

2) coherence spaces and stable (continuous) maps

3) observably sequential algorithms as in [CCF94]

by a theorem of A. Pitts (see [Pit96]) there exists a unique subset P of D such that $t \in P$ iff $t(\vec{s}) = \bot$ for all $\vec{s} \in P^\omega$. Such a P qualifies as a set PL of proof-like terms since P is closed under application, contains all elements definable in untyped λ-calculus and we also have $\mathsf{cc} \in P$.

For later use we remark that the identity map on D is represented by $\mathsf{i} \in P$ with $\mathsf{i}(t.\vec{s}) = t(\vec{s})$.

3 Some triposes induced by (D, P)

Since P is a subpca of the pca D we may consider the *relative realizability* topos $\mathcal{E} = \mathbf{RT}(D, P)$ induced by the tripos $\mathscr{P}$ over **Set** where for a set I the fibre $\mathscr{P}^I$ is the preorder $\big(\mathcal{P}(D)^I, \vdash_I\big)$ with $\phi \vdash_I \psi$ iff $\exists t \in P.\forall i \in I.\forall s \in \phi_i.\, ts \in \psi_i$ and for $u : J \to I$ reindexing along u is given by precomposition with u. For the set $\Sigma_{\mathscr{P}}$ of propositions of $\mathscr{P}$ we may take $\mathcal{P}(D)$ and for the *truth predicate* on $\Sigma_{\mathscr{P}}$ we may take $\mathrm{id}_{\mathcal{P}(D)}$.

Notice that $\Sigma_{\mathscr{P}}$ contains an "intermediate" truth value $U = \{\top_D\}$ which is neither equivalent to $\bot_{\Sigma_{\mathscr{P}}} = \emptyset$ nor to $\top_{\Sigma_{\mathscr{P}}} = D$. Moreover, in $\mathbf{RT}(D, P)$ the proposition $U = \{\top_D\}$ is equivalent to $U \vee \neg U$ (since $\neg U = \emptyset$) but not to D. Thus $U \vee \neg U$ does not hold in $\mathbf{RT}(D, P)$ for which reason the topos $\mathbf{RT}(D, P)$ is not boolean. However, the truth value U gives rise to the (Lawvere-Tierney) topology j_U on $\Sigma_{\mathscr{P}} = \mathcal{P}(D)$ which is defined as $j_U(A) = (A \to U) \to U$ for $A \in \mathcal{P}(D)$. We may form the full subtripos $\mathscr{P}_U$ of $\mathscr{P}$ consisting of j_U-closed predicates, i.e. $\phi \in \mathcal{P}(D)^I$ with $j_U \circ \phi \vdash_I \phi$. Since $j_U = \neg_U \circ \neg_U$ with $\neg_U A = A \to U$ the fibres of $\mathscr{P}_U$ are all boolean. We write $\mathcal{E}_U = \mathbf{RT}(D, P)_U$ for the ensuing boolean subtopos of $\mathcal{E} = \mathbf{RT}(D, P)$.

As described in the previous section $P \subseteq D$ gives rise to a classical realizability structure with *pole* $\perp\!\!\!\perp = \{\langle t, \vec{s}\rangle \mid t(\vec{s}) = \top\}$. We write $\mathcal{E}_{\perp\!\!\!\perp} = \mathbf{RT}(D, P)_{\perp\!\!\!\perp}$ or rather simply $\mathcal{K}$ for the ensuing classical realizability topos which is induced by the full subtripos $\mathscr{P}_{\mathcal{K}}$ of $\mathscr{P}$ consisting of those predicates $\phi \in \mathcal{P}(D)^I$ which factor through $\Sigma_{\mathcal{K}} = \{A \in \mathcal{P}(D) \mid A^{\perp\!\!\!\perp\perp\!\!\!\perp} = A\}$. We show now that

Lemma 3.1 *$\mathscr{P}_{\mathcal{K}}$ is equivalent to $\mathscr{P}_U$.*

Proof: First recall that on $\mathcal{P}(D)$ implication is given by $A \to B = \{t \in D \mid \forall s \in A.\, ts \in B\} = \{t \in D \mid \forall s \in A.\, \lambda\vec{r}.t(s.\vec{r}) \in B\}$ from which it follows that $\Sigma_{\mathcal{K}}$ is an exponential ideal in $\mathcal{P}(D)$, i.e. $A \to B$ is in $\Sigma_{\mathcal{K}}$ whenever B is in $\Sigma_{\mathcal{K}}$. Since $U \in \Sigma_{\mathcal{K}}$ the map j_U sends $\mathcal{P}(D)$ to $\Sigma_{\mathcal{K}}$. Thus, postcomposition with j_U gives rise to a tripos morphism from $\mathscr{P}$ to $\mathscr{P}_{\mathcal{K}}$ left adjoint to the inclusion of tripos $\mathscr{P}_{\mathcal{K}}$ into the tripos $\mathscr{P}$ (as induced by $\Sigma_{\mathcal{K}} \subseteq \mathcal{P}(D)$). Since $A \to j_U(A)$ is uniformly realized by $\eta = \lambda x.\lambda p.px \in P$ and for $A \in \Sigma_{\mathcal{K}}$ the implication $j_U(A) \to A$ is uniformly realized by $\mathsf{cc} \in P$ the adjunction above between $\mathscr{P}$ and $\mathscr{P}_U$ restricts to an equivalence between $\mathscr{P}_U$ and $\mathscr{P}_{\mathcal{K}}$.[1] □

Thus $\mathcal{K} = \mathbf{RT}(D, P)_{\perp\!\!\!\perp}$ and $\mathbf{RT}(D, P)_U$ are equivalent boolean subtoposes of the relative realizability topos $\mathcal{E} = \mathbf{RT}(D, P)$ which itself is not boolean. We write $i : \mathcal{K} \hookrightarrow \mathcal{E}$ for the corresponding injective geometric morphism. Its inverse image part $i^* : \mathcal{E} \to \mathcal{K}$ (sheafification) is given by postcomposition with j_U. Its (right adjoint) direct image part $i_* : \mathcal{K} \to \mathcal{E}$ is nontrivial. As described in [vO08] it sends an object X in $\mathcal{K}$ to $S(X)$, the object of singleton predicates on X in $\mathcal{K}$ considered as an object of $\mathcal{E}$.

For convenience we explicitate a bit the logical structure of the triposes introduced above.

[1] *Question* We know that $j_U(A) \to A^{\perp\!\!\!\perp\perp\!\!\!\perp}$ is realized by cc uniformly in $A \in \mathcal{P}(D)$. But is the reverse implication also realizable uniformly in A?

For $A, B \in \mathcal{P}(D)$ implication in $\mathscr{P}$ is given by $A \to B = \{t \in D \mid \forall s \in A. ts \in B\}$. Since the local operator j_U commutes with this implication it also works for $\mathscr{P}_U$. Looking a bit closer one sees that this holds also for $\mathscr{P}_\mathcal{K}$ since if A and B are biorthogonally closed then $A \to B = \{t \in D \mid \forall s \in A. \forall \vec{r} \in B^{\perp\!\!\perp}. t(s.\vec{r}) = \top\} = \{s.\vec{r} \mid s \in A, \vec{r} \in B^{\perp\!\!\perp}\}^{\perp\!\!\perp}$ and thus is biorthogonally closed.

For a set I universal quantification $\forall_I$ along the terminal projection $I \to 1$ is given by intersection, i.e. $\forall_I(\phi) = \bigcap_{i \in I} \phi_i$. Since $\forall_I(\phi \to U) = \big(\bigcup_{i \in I} \phi_i\big) \to U$ it is immediate that $\forall_I$ restricts to $\mathscr{P}_U$. This applies also to $\mathscr{P}_\mathcal{K}$ since $\forall_I(\phi) = \bigcap_{i \in I} \phi_i = \bigcap_{i \in I} \phi_i^{\perp\!\!\perp\perp\!\!\perp} = \big(\bigcup_{i \in I} \phi_i^{\perp\!\!\perp}\big)^{\perp\!\!\perp}$ for which reason $\forall_I(\phi)$ is biorthogonally closed. Universal quantification along arbitrary maps $u : J \to I$ in **Set** is given by

$$\forall_u(\phi)_i = \forall_J(\lambda j{\in}J. leq(u(j), i) \to \phi_j)$$

where *leq* stands for Leibniz equality.

Recall that Leibniz equality on set I is defined as

$$leq_I(i, j) = \bigcap_{p \in \Sigma^I} p(i) \to p(j)$$

where Σ refers to the Σ of the respective tripos. For the tripos $\mathscr{P}$ Leibniz equality on a set I is given by $leq_I(i, j) = \{\mathsf{i} \mid i = j\}$. Obviously, the predicate leq_I is equivalent to the predicate eq_I defined as $eq_I(i, j) = \{d \in D \mid i = j\}$. This observation is useful for obtaining a simple description of equality predicates for the tripos $\mathscr{P}_U$ since they are of the form $j_U \circ eq_I$. Notice that $j_U(\emptyset) = (\emptyset \to U) \to U = D \to U = U = \{\top_D\}$ and $j_U(D) = (D \to U) \to U = U \to U = \{d \in D \mid d\top_D = \top_D\} = \{d \in D \mid \forall \vec{s} \in D^\omega. d(\top_D.\vec{s}) = \top\} = \{\top_D\} \cup \uparrow\bar{0}$ where $\bar{0}$ is the least element of D sending $\top_D.\bot_D^\infty$ to $\top$. Thus, for $\mathscr{P}_U$ equality on I is given by $eq_I(i, j) = \{\top_D\} \cup \uparrow\{\bar{0} \mid i = j\}$. A different but equivalent implementation of equality on I for $\mathscr{P}_U$ is given by $eq_I(i, j) = \{\top_D\} \cup \{\mathsf{i} \mid i = j\}$ since there is a least $r \in P$ with $r\bot_D = \bot_D$, $r\top_D = \top_D$ and $rd = \mathsf{i}$ for $d \sqsupseteq \bar{0}$.

Since $\mathscr{P}_\mathcal{K}$ is equivalent to its subtripos $\mathscr{P}_U$ the above considerations apply to $\mathscr{P}_\mathcal{K}$ as well.

4 Nothing new in case of Scott domains

In a talk in Chambery in June 2012 [Kri12] Krivine has shown that a classical realizability model is a forcing model iff it validates the sentence[2] $\forall x^{\beth 2}(x \neq 0, x \neq 1 \to \bot)$, i.e. iff there exists a proof-like term realizing $|\top, \bot \to \bot| \cap |\bot, \top \to \bot|$. He has shown that from such a realizer one can construct a proof-like term Φ such that $\Phi \in |A|$ whenever $|A|$ contains some proof-like term.

This applies in particular to the realizability structures as described in section 2 where $|\top| = D$ and $|\bot| = \{\top_D\}$. Obviously, in this case $t \in P$ realizes $|\top, \bot \to \bot| \cap |\bot, \top \to \bot|$ iff $t\top_D s = \top_D = ts\top_D$ for all $s \in D$. But since $t \in P$

[2]in our terminology this means that $\forall x{:}2.(eq_2(x, 0) \vee eq_2(x, 1))$ holds in the tripos $\mathscr{P}_\mathcal{K}$

entails $t\bot_D\bot_D \neq \top_D$ this would give rise to a morphism $\vee : \Sigma \times \Sigma \to \Sigma$ with $u \vee v = \bot$ iff $u = v = \bot$ which does not exists in stable domain theory. However, in Scott domains such a morphism does exist ("parallel or") and allows one to construct an element of P realizing $|\top, \bot \to \bot| \cap |\bot, \top \to \bot|$. Moreover, in the case of Scott domains the classical realizability model induced by D and P is not only a forcing model but it is actually equivalent to the "ground model" **Set** as we show next.

Since P is Scott closed and closed under binary suprema it contains a greatest element $\Phi = \bigsqcup P$. Obviously, we have $\Phi(\vec{s}) = \bot$ iff $\vec{s} \in P^\omega$. Thus, a proposition A holds in the ensuing realizability model (i.e. $|A| \cap P \neq \emptyset$) iff $\Phi \in |A|$ (since $|A| = ||A||^{\perp\!\!\!\perp}$ is upward closed). Now for propositions A and B we have

$$
\begin{array}{l}
\Phi \in |A \to B| \quad \text{iff} \\
\forall t \in |A| \forall \vec{s} \in ||B||\ \Phi(t.\vec{s}) = \top \quad \text{iff} \\
\forall t \in |A| \forall \vec{s} \in ||B||\ t \notin P \vee \vec{s} \notin P^\omega \quad \text{iff} \\
\forall t \in |A| \forall \vec{s} \in ||B||\ t \in P \Rightarrow \vec{s} \notin P^\omega \quad \text{iff} \\
\forall t \in |A|\ \big(t \in P \Rightarrow \forall \vec{s} \in ||B||\ \vec{s} \notin P^\omega\big) \quad \text{iff} \\
\forall t \in |A|\ \big(t \in P \Rightarrow \Phi \in |B|\big) \quad \text{iff} \\
\big(\exists t \in P\ t \in |A|\big) \Rightarrow \Phi \in |B| \quad \text{iff} \\
\Phi \in |A| \Rightarrow \Phi \in |B|
\end{array}
$$

i.e. $A \to B$ holds iff from validity of A follows validity of B. Thus the ensuing classical realizability model is a 2-valued forcing model, i.e. coincides with the ground model **Set**.

The situation changes dramatically if one solves the domain equation for D in a category not admitting $\vee : \Sigma \times \Sigma \to \Sigma$ as e.g. the category **Coh** of coherence spaces and stable maps (see [GLT89]), the category **OSA** of observably sequential algorithms (see [CCF94]) or a category of HON games and innocent algorithms. Let us look more closely at the example of $D = \Sigma^{D^\omega}$ in **Coh** in which $||\bot|| = D^\omega$ and $||\top|| = \emptyset$ and accordingly $|\bot| = \{\top_D\}$ and $|\top| = D$. Now if $f \in |\top, \bot \to \bot| \cap |\bot, \top \to \bot|$ then $f\top_D\bot_D = \top_D = f\bot_D\top_D$ and thus, by stability of f, also $f\bot_D\bot_D = \top_D$ from which it follows that $f \notin P$. Thus the ensuing classical realizability model cannot be a forcing model (sheaves over a complete Boolean algebra) and, accordingly, is in particular different from the ground model **Set**.

5 Bifree Solution of $D = \Sigma^{D^\omega}$ in Coh

Let V be the least set with $V = \mathcal{P}_{\mathsf{fin}}(\omega \times V)$. If $\alpha \in V$ and $n \in \omega$ we write α_n for the set $\{\beta \in V \mid \langle n, \beta\rangle \in \alpha\}$. By recursion on $n \in \omega$ we will define a sequence of coherence spaces $D_n = \big(|D_n|, \frown_n\big)$ with $|D_n| \subseteq V$ such that

(1) $|D_n| \subseteq |D_{n+1}|$

(2) for $\alpha, \beta \in |D_n|$ we have $\alpha \frown_n \beta$ iff $\alpha \frown_{n+1} \beta$

(3) $\alpha \asymp_n \beta$ iff $\alpha \cup \beta \in D_n$

(4) $\alpha \frown_n \beta$ iff $\alpha \cup \beta \in D_n$ implies $\alpha = \beta$.

For getting the construction of the D_n right it is usful to recall that coherence spaces and linear continuous maps between them give rise[3] to a model of linear logic and that $\Sigma^{D^\omega} = !(D^\omega) \multimap \bot = (!(D^\omega))^\perp$. We put $|D_0| = \emptyset$, i.e. D_0 is the terminal object in **Coh**. Notice that (3) and (4) vacuously hold for D_0. For the induction step we put $D_{n+1} = \Sigma^{D_n^\omega} = \big(!(D_n^\omega)\big)^\perp$ as suggested by $\Sigma^{D^\omega} = !(D^\omega) \multimap \bot$. Thus, the web $|D_{n+1}|$ of D_{n+1} consists of all $\alpha \in V$ such that for all $k \in \omega$ it holds that $\alpha_k \in D_n$, i.e. $\beta \frown_n \gamma$ for all $\beta, \gamma \in \alpha_k$, since $|D_{n+1}|$ is the web of $!(D_n^\omega)$ and for this coherence space we have $\alpha \frown \beta$ iff $\alpha \cup \beta \in |D_{n+1}|$. Thus, for defining its orthogonal D_{n+1} we put $\alpha \frown_{n+1} \beta$ iff $\alpha \cup \beta \in |D_{n+1}|$ implies $\alpha = \beta$. Conditions (3) and (4) hold for D_{n+1} by construction since they hold for D_n by induction hypothesis. We write D for the coherence space where $|D| = \bigcup_{n \in \omega} |D_n|$ and $\frown$ is the union of the $\frown_n$.

Actually, one can avoid any explicit reference to the levels D_n and inductively define $|D|$ as the least subset of V with $\alpha \in |D|$ whenever $\forall n{\in}\omega.\, \alpha_n \subseteq |D| \wedge \forall \beta, \gamma \in \alpha_n.\, \beta \frown \gamma$ where $\beta \frown \gamma$ stands for $\beta \cup \gamma \in |D| \Rightarrow \beta = \gamma$. Notice that $|D|$ is closed under subsets and we have $\alpha \asymp \beta$ iff $\alpha \cup \beta \in |D|$.

Now we describe the realizability structure arising from D. The elements of $\Lambda_D = D$ are those $t \in \mathcal{P}(|D|)$ such that $\forall \alpha, \beta \in t.\, \alpha \frown \beta$, i.e. *antichains* in the poset $(|D|, \subseteq)$. The evaluation map $D \times D^\omega \to \Sigma$ is defined as follows: for $t \in D$ and $\vec{s} \in D^\omega$ we have $t(\vec{s}) = \top$ (notation $t \star \vec{s} \in \perp\!\!\!\perp$) iff $\exists \alpha \in t. \forall n \in \omega.\, \alpha_n \subseteq s_n$. With an $\vec{s} \in D^\omega$ one may associate the set $I_{\vec{s}} = \{\alpha \in |D| \mid \{\alpha\}(\vec{s}) = \top\} = \{\alpha \in |D| \mid \forall n \in \omega.\, \alpha_n \subseteq s_n\}$. Sets of this form can be characterized as downward closed ideals in $|D|$, i.e. subsets of D which are closed under subsets and finite unions. Any such ideal I is equal to $I_{\vec{s}}$ for a unique $\vec{s} \in D^\omega$ which is given by $s_n = \bigcup_{\alpha \in I} \alpha_n$. Writing Π_D for the set of downward closed ideals in $(|D|, \subseteq)$ for $t \in \Lambda_D$ and $\pi \in \Pi_D$ we have $t \star \pi \in \perp\!\!\!\perp$ iff $t \cap \pi \neq \emptyset$.

For exhibiting in a concrete way the remaining operations of the realizability structure induced by $D = \Sigma^{D^\omega}$ we have to introduce some notation. For a finite $a \in D$ and $\alpha \in |D|$ we write $a.\alpha$ for $(\{0\} \times a) \cup \{\langle n+1, \beta \rangle \mid \langle n, \beta \rangle \in \alpha\}$. For $t \in \Lambda_D$ and $\pi \in \Pi_D$ let $t.\pi = \{a.\alpha \mid a \subseteq_{\mathsf{fin}} t, \alpha \in \pi\}$. For $t, s \in \Lambda_D$ let $ts = \{\alpha \in |D| \mid \exists a \subseteq_{\mathsf{fin}} s.\ a.\alpha \in t\}$. For $f \in \mathbf{Coh}(D, D)$ let $\mathsf{fun}(f) = \{a.\alpha \mid (a, \alpha) \in \mathsf{tr}(f)\}$ where $\mathsf{tr}(f)$ is the trace of f, i.e. the set of all pairs (a, α) s.t. $a \in D$ is finite and $\alpha \in f(a)$ and for all $b \subseteq a$ from $\alpha \in f(b)$ it follows that $a = b$. Using fun we define $\lambda x.t = \mathsf{fun}(a \mapsto t[a/x])$. For $\pi \in \Pi_D$ we put $\mathsf{k}_\pi = \{\widehat{\alpha} \mid \alpha \in \pi\}$ where $\widehat{\alpha} = \{\langle 0, \alpha \rangle\}$ for $\alpha \in |D|$. We define $\mathsf{cc} = \big\{\{\{\widehat{\alpha}_1, \ldots, \widehat{\alpha}_k\}.\alpha\}.(\alpha \cup \alpha_1 \cup \ldots \cup \alpha_k) \mid \alpha \cup \alpha_1 \cup \ldots \cup \alpha_k \in |D|\big\}$.

Finally, we have to define which elements of Λ_D we want to consider as proof-like objects. By recursion on $\alpha \in |D|$ we define $|\alpha| \in \{0, 1\}$ as $|\alpha| = 1$ iff $\exists n \in \omega. \exists \beta \in \alpha_n.\, |\beta| = 0$. Thus $|\alpha| = 1$ iff α does not raise any error itself. Accordingly, we define the subset P of *proof-like* objects of D as $\{a \in D \mid \forall \alpha \in a.\, |\alpha| = 1\}$.

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[3]actually this model was the *source* of linear logic!

5.1 Some useful retractions in P

Let h_n be the sequence of subidentical retractions of D where $h_0 = \lambda f{:}D.\bot_D$ and $h_{n+1} = \Sigma^{h_n^\omega}$. Notice, that all h_n are elements of P since id_D is in P and $h_n \sqsubseteq \mathrm{id}_D$. Obviously, we have $\mathrm{id}_D = \bigsqcup_{n\in\omega} h_n$ but the images of the h_n typically contain elements which are not finite. Notice that the image of h_n is D_n for all $n \in \omega$.

There is also a subidentical retraction $r_P \in P$ sending $a \in D$ to $r_P(a) = \{\alpha \in a \mid |\alpha| = 1\}$. Obviously, the image of r_P is precisely P and $r_P(a)$ is the greatest element of P below a.

5.2 D is universal for countably based coherence spaces

To give an impression of the complexity of D we show that it contains every countably based coherence space via a stably continuous embedding/projection pair (see e.g. [AL91]). First recall that in Th. 2.4.2.9 of [AL91]) it has been shown that every coherence space X with countable web can be embedded into $!\mathbb{T}^\omega$ via a stably continuous embedding/projection pair (where $\mathbb{T}$ is the coherence space whose web consists of two incoherent tokens thought of as boolean values). Thus, the coherence space $X^\bot$ can be embedded into $(!\mathbb{T}^\omega)^\bot = \Sigma^{\mathbb{T}^\omega}$. Accordingly, all coherence spaces with countable web can be embedded into $\Sigma^{\mathbb{T}^\omega}$. Since $\mathbb{T}$ can be embedded into D the coherence space $\Sigma^{\mathbb{T}^\omega}$ can be embedded into Σ^{D^ω} and thus into D.

5.3 Antichains in Coherence Spaces

Let X be a coherence space. An *antichain* in X is a subset A of X such that $a, b \in A$ are equal whenever they are coherent (i.e. $a \cup b \in X$). We may order antichains in X "à la Smyth" as follows

$$A \leq_S B \quad \text{iff} \quad \forall y \in B.\exists x \in A.\ x \sqsubseteq y$$

i.e. $A \leq_S B$ iff $\uparrow A \supseteq \uparrow B$. This suggest to consider antichains as upward closed subsets C of X such that for the set $\min(C)$ of minimal elements of C it holds that

(1) $C \subseteq \uparrow\min(C)$ and

(2) coherent elements of $\min(C)$ are equal.

Under this view antichains may be considered as disjoint unions of *cones*, i.e. sets of the form $\uparrow x$ for some $x \in X$. We write $\mathscr{A}(X)$ for the set of antichains considered as upward closed subsets of X satisfying conditions (1) and (2) and consider it partially ordered by reverse subset inclusion. One can show that

Theorem 5.1 *$\mathscr{A}(X)$ is a complete lattice when ordered by $\supseteq$.*

Proof: Let $(C_i)_{i\in I}$ be a family of antichains in X. We show that its intersection $D := \bigcap_{i\in I} C_i$ is again an antichain from which it is immediate that D is the supremum of the C_i w.r.t $\supseteq$.

Obviously, the set D is upwards closed. For $x \in D$ and $i \in I$ let x_i be the unique element of $\min(C_i)$ with $x_i \sqsubseteq x$. Since $(x_i)_{i\in I}$ is bounded by x its supremum $m(x)$ exists. It is easy to see that $x \sqsupseteq m(x) \in \min(D)$. Thus D validates condition (1). For showing condition (2) suppose $x, y \in \min(D)$ have an upper bound. Then for all $i \in I$ we have $x_i \frown y_i$ and thus $x_i = y_i$ from which it follows that $m(x) = m(y)$ and thus $x = y$ as desired. □

An important class of antichains in X are those of the form $p^{-1}(\top)$ for some $p \in \Sigma^X$. Via trace they correspond to those $U \in \mathscr{A}(X)$ for which all elements of $\min(U)$ are compact elements of X. We write $\mathscr{A}_0(X)$ for this class of antichains in X. For every $C \subseteq X$ we may consider the antichain

$$\overline{C} = \bigcap\{U \in \mathscr{A}_0(X) \mid U \supseteq C\}$$

which, obviously, contains C as a subset. It is easy to see that $C \mapsto \overline{C}$ is a closure operator on $\mathcal{P}(X)$ since $\overline{C}$ is the intersection of all stably open subsets of X which contain C as a subset.[4]

For $X = D^\omega$ and $C \subseteq X$ we have $C^{\perp\!\!\!\perp} = \{t \in \Sigma^X \mid C \subseteq t^{-1}(\top)\}$ and thus $\overline{C} = C^{\perp\!\!\!\perp\perp\!\!\!\perp}$. Notice that the minimal elements of $C^{\perp\!\!\!\perp}$ w.r.t. the stable order are those $t \in D = \Sigma^{D^\omega}$ for which every element of $\mathsf{tr}(t)$ is below some element of C. For $t \in C^{\perp\!\!\!\perp}$ the unique minimal element $m(t)$ in $C^{\perp\!\!\!\perp}$ below t is characterized as follows: $e \in \mathsf{tr}(m(t))$ iff $e \in \mathsf{tr}(t)$ and $e \sqsubseteq x$ for some $x \in C$.

Since the infimum operation $\sqcap : \Sigma\times\Sigma \to \Sigma$ is stably continuous for $t_1, t_2 \in D$ we have $t_1 \sqcap t_2 \in D$. Obviously, we have $(t_1 \sqcap t_2)^{-1}(\top) = t_1^{-1}(\top) \cap t_2^{-1}(\top)$. Thus $C^{\perp\!\!\!\perp}$ is not only an antichain in D but it is also closed under $\sqcap$ and contains $\lambda x{:}X.\top$ as an element. It is an interesting but difficult problem to characterize those antichains A in D which are of the form $C^{\perp\!\!\!\perp}$ for some $C \subseteq X$. Well, it are those $A \subseteq D$ for which $A = A^{\perp\!\!\!\perp\perp\!\!\!\perp}$. But is there a more elementary combinatorial characterization of biorthogonally closed subsets of D? Such a characterization might be helpful for answering the question whether for any biorthogonally closed subset A of D either A or its negation $\neg_U A$ is inhabited by an element of P, i.e. whether $\mathcal{K}$ is 2-valued.

6 Exploring the structure of $\mathcal{E}$ and $\mathcal{K}$

We have seen that $\mathcal{K}$ is equivalent to **Set** when constructed from the bifree solution of $D = \Sigma^{D^\omega}$ in Scott domains. But something new arises when we start from the solution of this domain equation in **Coh**. We start now exploring this

[4]If X is a Scott domain then intersections of open subsets of X are just upward closed subsets of X. Alas, such an easy characterizations is not available for intersections of stably open subsets of a coherence space X.

new territory. Some attention will also be payed to the inituitionistic variant $\mathcal{E}$ in which computation is much easier than in its full subcategory $\mathcal{K}$ of j_U-sheaves.

For every $n \in \mathbb{N}$ let $\bar{n}$ be the unique element of D with $\bar{n}(\vec{s}) = \top$ iff $s_n = \top_D$. A "hardwired" version of this is $\bar{n} = \{\nu_n\}$ with $\nu_n = \{\langle n, \emptyset\rangle\} \in |D|$. From this it is obvious that the $\bar{n}$ are atoms of D and pairwise incoherent, i.e. $\nu_n \smile \nu_m$ iff $n \neq m$. Obviously, we have $\bar{n} \in P$ since $|\nu_n| = 1$.

In $\mathcal{E} = \mathbf{RT}(D, P)$ a natural numbers object is given by the assembly $N_{\mathcal{E}}$ with underlying set $\mathbb{N}$ and $||n||_{N_{\mathcal{E}}} = \{\bar{n}\}$. Similarly, the object $2_{\mathcal{E}}$ in $\mathcal{E}$ is given by the assembly with underlying set $2 = \{0, 1\}$ and $||k||_{2_{\mathcal{E}}} = \{\bar{k}\}$. The object $\Delta_{\mathcal{E}}(2)$ of $\mathcal{E}$ is given by the assembly with underlying set 2 and $||k||_{\Delta_{\mathcal{E}}(2)} = D$.

The corresponding objects $N_{\mathcal{K}}$, $2_{\mathcal{K}}$ and $\Delta_{\mathcal{K}}(2)$ in $\mathcal{K}$ are obtained from $N_{\mathcal{E}}$, $2_{\mathcal{E}}$ and $\Delta_{\mathcal{E}}(2)$ in $\mathcal{E}$ by *sheafification* (denoted as i^*), i.e. by postcomposing the respective equality predicates with j_U. But since j_U is a bit complex we are looking for somewhat simpler isomorphic copies of these objects in $\mathcal{K}$.

Since $j_U(\emptyset) = \{\top_D\}$ and $j_U(D) = \{\top_D\} \cup {\uparrow}\{\bar{0}\}$ for every set I the object $\Delta_{\mathcal{K}}(I)$ of $\mathcal{K}$ has underlying set I and equality predicate $[\![i \sim_{\Delta_{\mathcal{K}}(I)} j]\!] = \{\top_D\} \cup {\uparrow}\{\bar{0} \mid i = j\}$.

Next we determine $i^* N_{\mathcal{E}}$, the natural numbers object of $\mathcal{K}$ obtained by sheafifying the natural numbers object $N_{\mathcal{E}}$ of $\mathcal{E}$. The underlying set of $i^* N_{\mathcal{E}}$ is $\mathbb{N}$ and its equality predicate is given by $[\![n \sim_{i^* N_{\mathcal{E}}} m]\!] = j_U\big([\![n \sim_{N_{\mathcal{E}}} m]\!]\big)$. The following lemma exhibits an object $N_{\mathcal{K}}$ which in $\mathcal{K}$ is isomorphic to $i^* N_{\mathcal{E}}$ but simpler to describe and simpler to manipulate.

Lemma 6.1 *Let $N_{\mathcal{K}}$ be the object of $\mathcal{K}$ with underlying set $\mathbb{N}$ and equality predicate $[\![n \sim_{N_{\mathcal{K}}} m]\!] = \{\top_D\} \cup {\uparrow}\{\bar{n} \mid n = m\}$. In $\mathcal{K}$ the object $N_{\mathcal{K}}$ is isomorphic to $i^* N_{\mathcal{E}}$ and thus a natural numbers object in $\mathcal{K}$*

Proof: For showing the desired isomorphism it suffices to exhibit elements of P realizing the logical equivalence of $[\![n \sim_{N_{\mathcal{K}}} m]\!]$ and $j_U\big([\![n \sim_{N_{\mathcal{E}}} m]\!]\big)$ uniformly in n and m.

First notice that $[\![n \sim_{N_{\mathcal{K}}} n]\!] = \{\top_D\} \cup {\uparrow}\bar{n} = \{\vec{s} \in D^\omega \mid s_n = \top_D\}^{\perp\!\!\perp} \in \Sigma_{\mathcal{K}}$. Next we determine $j_U(\{\bar{n}\}) = (\{\bar{n}\} \to U) \to U$ for $n \in \mathbb{N}$. Observe that $\{\bar{n}\} \to U = \{d \in D \mid d\bar{n} = \top_D\} = \{d \in D \mid \forall \vec{s} \in D^\omega.\, d(\bar{n}.\vec{s}) = \top\}$. Thus, we have $j_U(\{\bar{n}\}) = \{d \in D \mid \forall d' \in D.\, d'\bar{n} = \top_D \Rightarrow dd' = \top_D\}$.

First we show that $\mathsf{cc} \in P$ realizes the implication $j_U\big([\![n \sim_{N_{\mathcal{E}}} m]\!]\big) \to [\![n \sim_{N_{\mathcal{K}}} m]\!]$ uniformly in n and m. If $n \neq m$ then $[\![n \sim_{N_{\mathcal{K}}} m]\!] = \{\top_D\} = j_U(\emptyset) = j_U\big([\![n \sim_{N_{\mathcal{E}}} m]\!]\big)$ and the claim follows since $\mathsf{cc}\top_D = \top_D$. Thus it suffice to show that cc realizes $j_U\big([\![n \sim_{N_{\mathcal{E}}} n]\!]\big) \to [\![n \sim_{N_{\mathcal{K}}} n]\!]$ for all n. For this purpose suppose $t \in j_U(\{\bar{n}\})$ and $\vec{s} \in D^\omega$ with $s_n = \top_D$. Then $\mathsf{k}(\vec{s}) \in \{\bar{n}\} \to U$ since for $\vec{r} \in D^\omega$ we have $\mathsf{k}(\vec{s})(\bar{n}.\vec{r}) = \bar{n}(\vec{s}) = \top$. Thus, we have $\mathsf{cc}(t.\vec{s}) = t(\mathsf{k}(\vec{s}).\vec{s}) = \top$ as desired since $t \in j_U(\{\bar{n}\})$ and $\mathsf{k}(\vec{s}) \in \{\bar{n}\} \to U$.

There is an $e \in P$ with $e\top_D = \top_D$ and $e\bar{n}d = d\bar{n}$ for all $n \in \mathbb{N}$ and $d \in D$. Obviously, such an e realizes $\{\top_D\} \to j_U(\emptyset)$. Moreover, for $n \in \mathbb{N}$ we have $e\bar{n} \in j_U(\{\bar{n}\})$ since if $d\bar{n} = \top_D$ then also $e\bar{n}d = d\bar{n} = \top_D$. Thus, since $j_U(\{\bar{n}\})$ is upward closed for every $d \sqsupseteq \bar{n}$ we have $e\bar{n} \sqsubseteq ed \in j_U(\{\bar{n}\})$.

Thus e realizes $\{\top_D\} \cup \uparrow\bar{n} \to j_U(\{\bar{n}\})$. Thus, we have shown that e realizes $[\![n \sim_{N_\mathcal{K}} m]\!] \to j_U\big([\![n \sim_{N_\mathcal{E}} m]\!]\big)$ uniformly in n and m. □

Similarly, one shows that in $\mathcal{K}$ the object $i^*2_\mathcal{E}$ is isomorphic to the object $2_\mathcal{K}$ with underlying set $2 = \{0,1\}$ and equality predicate $[\![i \sim_{2_\mathcal{K}} j]\!] = \{\top^D\} \cup \uparrow\{\bar{i} \mid i = j\}$. Since $\mathcal{K}$ is a boolean topos the truth value object $\Omega_\mathcal{K}$ is known to be isomorphic to $2_\mathcal{K}$. We do not know whether the object $2_\mathcal{K}$ has precisely two global elements, i.e. whether the topos $\mathcal{K}$ is 2-valued.[5]

Since $\mathcal{K}$ is a subtopos of $\mathcal{E}$ arising from the Lawvere-Tierney topology j_U on $\mathcal{E}$ there is an induced injective geometric morphism $i : \mathcal{K} \hookrightarrow \mathcal{E}$ whose inverse image part $i^* : \mathcal{E} \to \mathcal{K}$ we have already described. It is fairly simple since it is given by postcomposition with j_U. However, its right adjoint i_*, the direct image part of i, though full and faithful is not simply inclusion in the naive sense. As described e.g. in [vO08] it sends an object X of $\mathcal{K}$ to the object i_*X of $\mathcal{E}$ which is the object $S(X)$ of 'singleton predicates' on X in $\mathcal{K}$ considered as an object of $\mathcal{E}$. The underlying set of $S(X)$ is the set of all functions from $|X|$ to $\Sigma_\mathcal{K}$ where $|X|$ is the underlying set of X. The existence predicate $E_{S(X)}$ on $\Sigma_\mathcal{K}^{|X|}$ is given by

$$E_{S(X)}(A) = [\![\mathit{Pred}_X(A) \wedge \exists x{:}|X|.\forall y{:}|X|.A(y) \leftrightarrow x \sim_X y]\!]$$

where

$$\mathit{Pred}_X(A) = [\![\forall x{:}|X|.A(x) \to x \sim_X x \wedge (\forall y{:}|X|.x \sim_X y \to A(y))]\!]$$

and the equality predicate for $S(X)$ is given by

$$[\![A \sim_{S(X)} B]\!] = [\![E_{S(X)}(A) \wedge \forall x{:}|X|.A(x) \leftrightarrow B(x)]\!]$$

which finishes the description of $S(X)$. For the morphism part of S suppose $F : |X|\times|Y| \to \Sigma_\mathcal{K}$ represents a morphism from X to Y. Then the corresponding morphisms from $S(X)$ to $S(Y)$ is given by the $\Sigma_\mathcal{K}$-valued predicate $S(F) : \Sigma_\mathcal{K}^{|X|} \times \Sigma_\mathcal{K}^{|Y|} \to \Sigma_\mathcal{K}$ defined as

$$S(F)(A,B) = [\![E_{S(X)}(A) \wedge E_{S(Y)}(B) \wedge \forall x{:}|X|, y{:}|Y|.F(x,y) \leftrightarrow (A(x) \wedge B(y))]\!]$$

for $A \in \Sigma_\mathcal{K}^{|X|}$ and $B \in \Sigma_\mathcal{K}^{|Y|}$. Thus, though the inclusion of $\mathcal{K}$ into $\mathcal{E}$ via i_* preserves exponentials due to the complicated nature of i_* there is not much gain when computing the exponentials in the relative realizability topos $\mathcal{E}$.

Generally, since classical realizability toposes are boolean Ω is isomorphic to 2. Thus, since 2 is a subobject of N the exponential N^N contains $2^N \cong \mathcal{P}(N)$ as a subobject which explains why in general N^N is so complicated in classical realizability toposes. Maybe this is the reason why Krivine in his papers considers classical realizability models for classical second order logic or the classical set theory ZF which are both based on sets and not on functions. In

[5] But it can be shown that for *countable* $A \subseteq D$ either $A^{\perp\!\!\!\perp\perp\!\!\!\perp}$ or its negation are true in $\mathcal{K}$.

both settings functions appear only as a derived concept, namely as functional relations, i.e. particular sets.[6]

So far we do not know yet whether $\mathcal{K}$ is actually different from a forcing model. But it will follow from the results of the following subsection where we show that

$\mathcal{K}$ is not even a Grothendieck topos

Since there is no parallel-or in the realizability structure induced by $P \subseteq D$ it follows from Krivine's observation in [Kri12] that the object $\Delta_{\mathcal{K}}(2)$ is not isomorphic to $2_{\mathcal{K}}$. For this reason the tripos $\mathscr{P}_{\mathcal{K}}$ does not arise from a complete boolean algebra. But from this it does not follow yet that $\mathcal{K}$ is not equivalent to a forcing model, i.e. a localic boolean topos, since non-equivalent triposes might induce the same topos. But we will show now that $\mathcal{K}$ is not even a Grothendieck topos and thus *a fortiori* not a forcing model.

For this purpose we will proceed in two steps. First in Lemma 6.2 we will show that every Grothendieck subtopos of $\mathcal{K}$ is equivalent to **Set** and then in the subsequent Lemma 6.3 we will show that $\mathcal{K}$ is not equivalent to **Set**. It is then an immediate consequence of these two lemmas that

Theorem 6.1
$\mathcal{K}$ is not a Grothendieck topos and thus, in particular, not a forcing model.

The following considerations are necessary as preparation for the proofs of Lemma 6.2 and 6.3.

There is a geometric inclusion $\Pi_{\mathcal{E}} \dashv \Delta_{\mathcal{E}} : \mathbf{Set} \hookrightarrow \mathcal{E}$ where $\Pi_{\mathcal{E}}$ is given by $\mathcal{E}(U,-)$. The right adjoint $\Delta_{\mathcal{E}}$ sends set I to the object $\Delta_{\mathcal{E}}(I) = (I, eq_I)$ (see section 3) and $u : J \to I$ in **Set** to the morphism $\Delta_{\mathcal{E}}(u) : \Delta_{\mathcal{E}}(J) \to \Delta_{\mathcal{E}}(I)$ represented by the $\mathscr{P}$-predicate $eq_I(u(j), i)$ on $J \times I$. Notice that $\Delta_{\mathcal{E}}$ factors through $\mathbf{Asm}(P, D)$, the category of assemblies in $\mathbf{RT}(D, P)$, since $\Delta_{\mathcal{E}}(I)$ is isomorphic to the assembly with underlying set I and $||i|| = D$ for all $i \in I$. The restriction of the left adjoint $\Pi_{\mathcal{E}}$ to $\mathbf{Asm}(P, D)$ sends an assembly to its underlying set and a morphism to its underlying set-theoretic function. Notice that $\Pi_{\mathcal{E}} \dashv \Delta_{\mathcal{E}} : \mathbf{Set} \hookrightarrow \mathcal{E}$ is the least non-trivial subtopos of $\mathcal{E}$ induced by the double negation topology on $\mathcal{E}$.

We write $\bar{D}$ for the object of $\mathbf{Asm}(P, D)$ with underlying set D and $||t||_{\bar{D}} = \{t\}$ for $t \in D$. Obviously, the counit $\eta_{\bar{D}} : \bar{D} \to \Delta_{\mathcal{E}}\Pi_E\bar{D}$ is monic. If $j : \mathcal{F} \hookrightarrow \mathcal{E}$ is a nontrivial subtopos of $\mathcal{E}$ then the counit $\bar{D} \to j_*j^*\bar{D}$ of $j^* \dashv j_*$ at $\bar{D}$ factors along η_D via a subobject $j_*j^*\bar{D} \rightarrowtail \Delta_{\mathcal{E}}\Pi_E\bar{D}$ whose characteristic predicate χ_D is given by $\chi_D(t) = j_{\mathcal{F}}(\{t\})$ for $t \in D$ where $j_{\mathcal{F}}$ is the closure operator on $\mathcal{E}$ inducing the subtopos $\mathcal{F}$ of $\mathcal{E}$. Thus $j_*j^*\bar{D}$ is (isomorphic to) the assembly with underlying set D and $||t||_{j_*j^*\bar{D}} = j_{\mathcal{F}}(\{t\})$.

Now adapting an argument from [Joh13] we show that

[6] Of course, in case of second order logic he has to permit *function constants* on the underlying (countable) set of objects (usually identified with the set natural numbers).

Lemma 6.2
Every nontrivial Grothendieck subtopos $\mathcal{F}$ of $\mathcal{E}$ is equivalent to **Set***.*

Proof: Suppose $\mathcal{F}$ is a nontrivial Grothendieck subtopos of $\mathcal{E}$. We write $j : \mathcal{F} \hookrightarrow \mathcal{E}$ for the corresponding inclusion. Since $\mathcal{F}$ is a Grothendieck topos it has arbitrary copowers. We write $\Delta_{\mathcal{F}}(I)$ for the I-fold copower of $1_{\mathcal{F}}$, i.e. $\coprod_I 1_{\mathcal{F}}$. Notice that $C = \Delta_{\mathcal{F}}(D)$ and $j_*j^*\bar{D}$ are both assemblies. Since $\mathbf{Asm}(P, D)$ is an exponential ideal in $\mathbf{RT}(D, P)$ the exponential $(j_*j^*\bar{D})^C$ is an assembly, too, and, moreover, (isomorphic to) the D-fold product of $j_*j^*\bar{D}$. The underlying set of $(j_*j^*\bar{D})^C$ may be identified with the set of all functions from D to D since $\Pi_{\mathcal{E}}((j_*j^*\bar{D})^C) \cong \mathcal{E}(U, j_*j^*\bar{D})^C) \cong \mathcal{E}(C, (j_*j^*\bar{D})^U) \cong \mathcal{E}(U, j_*j^*\bar{D})^D \cong \mathbf{Set}(D, D)$.

Let $j_{\mathcal{F}}$ be the closure operator on $\mathcal{E}$ giving rise to the subtopos $\mathcal{F}$ of $\mathcal{E}$. The subobject $j_*j^*\bar{D} \rightarrowtail \Delta_{\mathcal{E}}\Pi_{\mathcal{E}}\bar{D}$ is classified by the predicate $\chi_D(t) = j_{\mathcal{F}}(\{t\})$. Obviously, the Grothendieck topos $\mathcal{F}$ is equivalent to **Set** iff χ_D is constantly true, i.e. there is a $t \in P$ with $t \in \bigcap_{t \in D} j_{\mathcal{F}}(\{t\})$.

For sake of contradiction suppose this were not the case. Then by axiom of choice on the meta-level there exists a (typically non-continuous) function $g : D \to D$ with $t \notin j_{\mathcal{F}}(\{g(t)\})$. For $t \in D$ let $s_t \in P$ be some realizer for the projection $\pi_t : (j_*j^*\bar{D})^C \to j_*j^*\bar{D} : h \mapsto h(t)$. Let $f : D \to D : t \mapsto g(s_t t)$ for which it obviously holds that $s_t t \notin j_U(\{f(t)\})$ for $t \in D$. Since $\Pi_{\mathcal{E}}((j_*j^*\bar{D})^C) \cong \mathbf{Set}(D, D)$ there is a $t \in D$ realizing f as an object of $(j_*j^*\bar{D})^C$. But then $s_t t \in j_U(\{f(t)\})$ which is impossible. □

Now for showing Theorem 6.1 it remains to prove that

Lemma 6.3
The topos $\mathcal{K}$ is not equivalent to **Set***.*

Proof: For sake of contradiction suppose that $\mathcal{K}$ is equivalent to **Set**. Then $i_*i^*\bar{D} \rightarrowtail \Delta_{\mathcal{E}}\Pi_{\mathcal{E}}\bar{D}$ is an isomorphism. But then the predicate χ_D on $\Delta_{\mathcal{E}}\Pi_{\mathcal{E}}\bar{D}$ is constantly true, i.e. there is an $s \in P$ with $s \in j_U(\{t\})$ for all $t \in D$. But this is impossible since already $j_U(\{\top_D\}) = (U \to U) \to U$ does not contain an element of P (since such an element t would map $\mathsf{i} \in (U \to U) \cap P$ to an element $t\mathsf{i} \in U \cap P = \emptyset$). □

7 $\mathcal{K}$ is a model of full first order arithmetic

Since $\mathcal{K}$ hosts a natural numbers object $N_{\mathcal{K}}$ it is most natural to ask how much of first order arithmetic holds in $\mathcal{K}$. First notice that *all* functions on $\mathbb{N}$ do exist as morphisms in $\mathcal{K}$. An arbitrary set-theoretic function $f : \mathbb{N} \to \mathbb{N}$ is represented as the morphism $f_{\mathcal{K}} : N_{\mathcal{K}} \to N_{\mathcal{K}}$ as given by the $\mathscr{P}_{\mathcal{K}}$-predicate $[\![f(n) \sim_{N_{\mathcal{K}}} m]\!]$ on $\mathbb{N} \times \mathbb{N}$ because there exists $t_f \in P$ with $t_f\bar{n} = \overline{f(n)}$ for all $n \in \mathbb{N}$. Equality of natural numbers will be interpreted as $[\![\cdot \sim_{N_{\mathcal{K}}} \cdot]\!]$. Propositional logical connectives will be interpreted as usual (see [Kri09]) but notice that

$|A{\to}B| = \{t \in D \mid \forall s \in |A|.\, ts \in |B|\}$.[7] Universal quantification over $N_{\mathcal{K}}$ is interpreted as

$$|\forall x.A(x)| = \bigcap_{n\in\mathbb{N}} [\![n \sim_{N_{\mathcal{K}}} n]\!] \to |A(n)|$$

which is coincidence with [Kri09] since the equivalence of $[\![n \sim_{N_{\mathcal{K}}} n]\!]$ and

$$\bigcap_{X\in\Sigma_{\mathcal{K}}^{\mathbb{N}}} X(0) \to \forall x(X(x) \to X(x{+}1)) \to X(n)$$

can be realized by an element of P. As usual existential quantification over $N_{\mathcal{K}}$ is interpreted as its second order encoding, i.e.

$$|\exists x.A(x)| = \bigcap_{X\in\Sigma_{\mathcal{K}}} \left(\bigcap_{n\in\mathbb{N}} ([\![n \sim_{N_{\mathcal{K}}} n]\!] \to |A(n)| \to X) \right) \to X$$

from which it follows that $\lambda f.f\bar{n}t \in |\exists x.A(x)|$ whenever $t \in |A(n)|$. Now we are ready to prove that

Theorem 7.1 *$\mathcal{K}$ validates all true sentences of first order arithmetic.*

Proof: Since $\mathcal{K}$ is boolean and classically every first order sentence is provably equivalent to a sentence in *prenex form*, i.e. a prefix of quantifiers followed by an equation between arithmetic terms, it suffices to show that all true arithmetic formulas in prenex form do hold in $\mathcal{K}$. We proceed by structural induction on the structure of arithmetical sentences in prenex form.

If $e_1 = e_2$ is a true arithmetical equation where both sides have value $n \in \mathbb{N}$ then $e_1 = e_2$ is realized by $\bar{n} \in P$.

Suppose $\forall x.A(x)$ is a true arithmetical sentence in prenex form. Then for all $n \in \mathbb{N}$ the sentence $A(n)$ is true and in prenex form. Thus, by induction hypothesis for every $n \in \mathbb{N}$ there is a $p_n \in P$ realizing $A(n)$. Then there exists a $t \in P$ with $t\top_D = \top_D$ and $t\bar{n} = p_n$ for all $n \in \mathbb{N}$. Obviously t realizes $\forall x.A(x)$.

Suppose $\exists x.A(x)$ is a true arithmetical sentence in prenex form. Then for some $n \in \mathbb{N}$ the sentence $A(n)$ is true and in prenex form. By induction hypothesis there is a $p \in P$ realizing $A(n)$ from which it follows that $\lambda f.f\bar{n}p \in P$ realizes $\exists x.A(x)$. $\square$

Thus, w.r.t. first order arithmetic sentences one cannot distinguish $\mathcal{K}$ from **Set**. But already at second order things get much more delicate since one does not even know whether every morphism $N_{\mathcal{K}} \to N_{\mathcal{K}}$ in $\mathcal{K}$ is induced by a map $\mathbb{N} \to \mathbb{N}$ in **Set**, i.e. whether for any functional relation F from $N_{\mathcal{K}}$ to $N_{\mathcal{K}}$ there exists function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall x, y{:}N_{\mathcal{K}}.F(x, y) \leftrightarrow f(x) \sim_{n_{\mathcal{K}}} y$ holds in $\mathcal{K}$. One easily sees that f is uniquely determined by F but the question rather is whether for all F such an f exists.

Actually, there is an even simpler question of this kind for which we do not know the answer so far, namely whether in $\mathcal{K}$ the natural numbers object $N_{\mathcal{K}}$

[7] As in [Kri09] we write $|A|$ for the interpretation of formula A.

has only "standard" global elements. More explictly, this means whether for any morphism $a : 1_\mathcal{K} \to N_\mathcal{K}$ in $\mathcal{K}$ there is an $n \in \mathbb{N}$ such that $a \sim_{N_\mathcal{K}} n$ holds in $\mathcal{K}$. The answer is definitely negative for boolean valued models $\mathsf{Sh}(B)$ when B is a complete boolean algebra with more than 2 elements. Since if $u \in B$ is different from 0_B and 1_B then so is $\neg u$ and one may cook up a "mixed" natural number a which is 0 on u and 1 on $\neg u$. We could come up with a similar "nonstandard" global element of $N_\mathcal{K}$ if $\Omega_\mathcal{K} = 2_\mathcal{K}$ were not 2-*valued*, i.e. if their existed an $u : 1 \to 2_\mathcal{K}$ for which $\mathcal{K}$ validates neither $u \sim_{2_\mathcal{K}} 0$ nor $u \sim_{2_\mathcal{K}} 1$ though it certainly validates the disjunction $u \sim_{2_\mathcal{K}} 0 \vee u \sim_{2_\mathcal{K}} 1$.

8 The object $\Delta_\mathcal{K}(2)$ is infinite

In [Kri12] J.-L. Krivine has shown that $\Delta_\mathcal{K}(2)$ does not contain any atoms (w.r.t. the order $\Delta_\mathcal{K}(\leq_2)$), i.e.

$$\forall x{:}\Delta_\mathcal{K}(2)\big(x \neq 0 \to \exists y{:}\Delta_\mathcal{K}(2)\, xy \neq 0 \wedge xy \neq x\big)$$

which by classical logic is equivalent to

$$\forall x{:}\Delta_\mathcal{K}(2)\big(\forall y{:}\Delta_\mathcal{K}(2)(xy \neq 0 \to xy \neq x \to \bot) \to x \neq 0 \to \bot\big)$$

For sake of completeness we recall Krivine's argument for which purpose we have to introduce a bit of machinery. For $I \subseteq_{\mathsf{fin}} \mathbb{N}$ let $\bar{I} \in D$ with $\bar{I}(\vec{s}) = \top$ iff $s_i = \top_D$ for all $i \in I$. Notice that $\bar{\emptyset} = \top_D$ and $\overline{\{n\}} = \bar{n}$. Obviously, we have

(1) $\top_D \sqsubseteq u$ iff $u \in |\top, \bot \to \bot| \cap |\bot, \top \to \top|$

(2) $u \Vdash \bot, \bot \to \bot$ iff $u \in \uparrow\{\bar{I} \mid \emptyset \neq I \subseteq \{0,1\}\}$.

Let $t \in D$ with $t\top_D = \top_D$ and $t\bar{I} = \bar{0}$ for nonempty subsets I of $\{0,1\}$. Then t realizes both

$$|\top, \bot \to \bot| \cap |\bot, \top \to \top|, \top \to \bot \qquad \text{and} \qquad (\bot, \bot \to \bot), \bot \to \bot$$

and thus t realizes $\forall x{:}\Delta_\mathcal{K}(2)\big(\forall y{:}\Delta_\mathcal{K}(2)(xy \neq 0 \to xy \neq x \to \bot) \to x \neq 0 \to \bot\big)$ as can be seen by case analysis on $x \in \{0,1\}$.

Thus, in $\mathcal{K}$ it holds that $\Delta_\mathcal{K}(2)$ is infinite. But it is not clear *a priori* whether $\Delta_\mathcal{K}(2)$ is also *Dedekind infinite*, i.e. whether the assertion

$$\exists f{:}\Delta_\mathcal{K}(2)^{N_\mathcal{K}}(\forall n, m{:}N_\mathcal{K}.f(n) \sim_{2_\mathcal{K}} f(m) \to n \sim_{N_\mathcal{K}} m)$$

holds in $\mathcal{K}$.[8] Actually, for quite some time we hoped that in $\mathcal{K}$ the object $\Delta_\mathcal{K}(2)$ would not be Dedekind infinite since this would have had the consequence

[8] See e.g. [Je73] for the construction of a model of ZF in which there exists a Dedekind finite set which is not finite. This cannot be achieved by forcing since forcing models all validate AC. One has to consider an appropriate group G of automorphism on an appropriate complete boolean algebra B and take the G-invariant part of the B-valued model.

that $\mathcal{K}$ does not validate *countable choice*.[9] The reason for this hope was that presumably there does not exist a monomorphism $N_{\mathcal{K}} \rightarrowtail \Delta_{\mathcal{K}}(2)$ in $\mathcal{K}$.

However, we will show that $\mathcal{K}$ does indeed validate *countable* and even *dependent choice*. From this it follows that $\mathcal{K}$ validates the assertion that there exists an injective function from $N_{\mathcal{K}}$ to $\Delta_{\mathcal{K}}(2)$ though presumably this existential statement is not witnessed by a global element of $\Delta_{\mathcal{K}}(2)^{N_{\mathcal{K}}}$, i.e. a proper monomorphism $N_{\mathcal{K}} \rightarrowtail \Delta_{\mathcal{K}}(2)$ in $\mathcal{K}$.

9 $\mathcal{K}$ validates countable and dependent choice

Though Krivine's classical realizability gives rise to models of the classical set theory ZF (as described in [Kri01]) it generally does *not* validate the full axiom of choice. Moreover, it is not known whether all classical realizability models for ZF validate the principles of dependent or at least countable choice. Though, unfortunately, so far we do not know any counterexample J.-L. Krivine strongly suspects that the answer to this question will be negative. In his opinion for realizing countable and dependent choice one has to extend his λ-calculus with control with new language constructs as described in [Kri03] where he adds a variant of LISP and Scheme's `quote` construct and shows how this may be used for realizing the above mentioned choice principles. But this method works only if the set Λ of "terms" is countable which is, obviously, not the case for the realizability structure arising from $D = \Sigma^{D^\omega}$ in **Coh** since D has the size of the continuum.

However, as known from work of C. Spector dating back to the early 60s one may use *bar recursion* for realizing classical choice principles. This approach has been applied fruitfully in "traditional" proof theory as described and discussed in U. Kohlenbach's monograph [Koh08]. However, Spector's original work and most of [Koh08] are based on Gödel's *Dialectica* interpretation and not on realizability. Thus, for our purposes the approach of U. Berger and P. Oliva in [BO05] (also discussed in [Koh08]) is a better starting point since it is based on modified realizability which can be adapted more easily to the case of classical realizability.[10]

In [BO05] it is shown that when starting from a model $\mathcal{M}$ of higher type arithmetic validating an appropriate form of bar recursion certain negative translations (where $\bot$ is replaced by arbitrary Σ^0_1-formulas) of classical choice principles admit a modified realizability interpretation by objects of $\mathcal{M}$. For this purpose in [BO05] they consider a 'modified bar recursor' whose analogue in our setting we will introduce next after some preliminary remarks.

First of all for a coherence space X we have to consider $X^* = \coprod_{n\in\mathbb{N}} X^n$, the set of *lists* of elements of X, which *per se* is not a coherence space since it lacks

[9] As remarked in [Je73] for any infinite set using countable choice one can prove quite straightforwardly the existence of an injective function from $\mathbb{N}$ into this set.

[10] This does not mean that methods based on Gödel's *Dialectica* interpretation are not more appropriate for the purposes of extracting programs and bounds from (classical) proofs as emphasized in [Koh08].

a least element. However, we say that a map f from X^* to a coherence space Y is *stable* iff for all $n \in \mathbb{N}$ the restriction of f to X^n is stable. Moreover, if Y is a coherence space then $X^* \to Y \cong \prod_{n\in\omega} X^n \to Y$ is a coherence space since the $X^n \to Y$ are coherence spaces. Alternatively, we may work in the slightly larger category $\omega\mathbf{dI}_c$ of coherently complete countably based dI-domains (see [AC98]) and stable continuous functions between them. We will have to consider stable functionals in the finite type hierarchy in **Coh** generated from Σ and D by $\to$, $(-)^\omega$ and $(-_1)^* \to (-_2)$. For every such type X we have to specify its subset PL_X of *proof-like* elements. Of course, for D we put $\mathsf{PL}_D = P$ and for Σ we put $\mathsf{PL}_\Sigma = \{\bot\}$. If X and Y are such types we put $\mathsf{PL}_{X\to Y} = \{f : X \to Y \mid \forall x \in \mathsf{PL}_X. f(x) \in \mathsf{PL}_Y\}$, $\mathsf{PL}_{X^\omega} = \mathsf{PL}_X^\omega$ and $\mathsf{PL}_{X^*} = \mathsf{PL}_X^*$, i.e. we extend PL à la logical relations.

Definition 9.1 *Given $Y : D^\omega \to \Sigma$ and $G : ((D \to \Sigma) \to \Sigma)^\omega$ in* **Coh** *let $\mathsf{BR}(Y, G)$ be the least stable function $\Psi : D^* \to \Sigma$ in* **Coh** *satisfying*

$$\Psi(s) = Y(s * \lambda n.G_{|s|}(\lambda x.\Psi(s * x)))$$

for all $s \in D^*$.

Obviously, the ensuing map $\mathsf{BR} : (D^\omega \to \Sigma) \to ((D \to \Sigma) \to \Sigma)^\omega \to \Sigma$ is stable and proof-like.

Notice that all the types built from D and Σ by $\to$, $(-)^\omega$ and $(-_1)^* \to (-_2)$ appear as retracts of D via proof-like maps. They form a typed pca realizability over which gives rise to a category equivalent to $\mathbf{RT}(D, P)$ as described on a more general level in [LS02]. This allows us to assume that realizers of particular propositions have particular types which often allows us to reason in a more intuitive way.

We often will have to refer to $\Omega_\mathcal{K}$ considered as an object of $\mathcal{E}$. This object has underlying set $\{A \in \mathcal{P}(D) \mid A^{\perp\!\!\!\perp\perp\!\!\!\perp} = A\}$ for which equality is given by logical equivalence[11]. Moreover, for objects X in $\mathcal{K}$ the exponential $\mathcal{P}_\mathcal{K}(X) = \Omega_\mathcal{K}^X$ is the same when taken in $\mathcal{E}$ and $\mathcal{K}$, respectively. Moreover, for X in $\mathcal{E}$ the map $\Omega_\mathcal{K}^{\eta_X} : \Omega_\mathcal{K}^{i_* i^* X} \to \Omega_\mathcal{K}^X$ (where $\eta_X : X \to i_* i^* X$ is the unit of $i^* \dashv i_*$ at X) is an isomorphism in $\mathcal{E}$. Accordingly, we will often write $\mathcal{P}_\mathcal{K}(X)$ for $\Omega_\mathcal{K}^X$ in $\mathcal{E}$.

9.1 Countable Choice holds in $\mathcal{K}$

Spector already observed that the negative translation of countable choice can be proved in any intuitionistic theory validating countable choice and the principle of *Double Negation Shift* (DNS) for formulas in the negative fragment. Like all relative realizability toposes $\mathcal{E} = \mathbf{RT}(D, P)$ validates countable and dependent choice. Thus, due to Spector's observation it suffices to show that $\mathcal{E}$ also validates an appropriate form of DNS.

[11] which is the same in $\mathcal{E}$ and $\mathcal{K}$ for propositions of this particular form

Lemma 9.1 *The topos $\mathcal{E}$ validates the principle*

$$\text{(DNS)} \qquad \forall B{:}\mathcal{P}_{\mathcal{K}}(N).\forall n.{\sim\sim}B(n) \to {\sim\sim}\forall n.B(n)$$

where ${\sim}A$ *stands for* $A \to U$*.*

Proof: Suppose $B \in \mathcal{P}(D)^\omega$ with $B(n)^{\perp\!\!\perp\perp\!\!\perp} = B(n)$ for all n, G of type $((D \to \Sigma) \to \Sigma)^\omega$ realize $\forall n.{\sim\sim}B(n)$ and Y of type $D^\omega \to \Sigma$ realize ${\sim}\forall n.B(n)$. Let $\Psi = \mathsf{BR}(Y, G)$. Using a variant of bar induction as described in [BO05] we will show now that $\Psi(\langle\rangle) = \top$ and thus realizes U.

We write $S(x, n)$ for $x \in B(n)$ and $P(s)$ for $\Psi(s) = \top$. We employ the abbreviations $s \in S \equiv \forall k < |s|\ s_k \in B(k)$ and $\alpha \in S \equiv \forall k\ \alpha_k \in B(k)$. By bar induction relativized to S (see [BO05] for details) for showing $P(\langle\rangle)$ it suffices to show that

(1) $\forall \alpha \in S \exists n\ P(\bar{\alpha}(n))$

(2) $\forall s \in S\big(\forall x(S(x, |s|) \to P(s * x))\big) \to P(s)$.

ad (1) : Suppose $\alpha \in S$, i.e. $\alpha(n) \in B(n)$ for all n. Then by assumption on Y we have $Y(\alpha) = \top$. Since Y is continuous there exists an n with $Y(\alpha) = Y(\bar{\alpha}(n)*\beta)$ for all β. Thus, we have $\Psi(\bar{\alpha}(n)) = \top$, i.e. $P(\bar{\alpha}(n))$ as desired.
ad (2) : Suppose $s \in S$ with $\forall x(S(x, |s|) \to P(s*x))$, i.e. $\forall x(x \in B(|s|) \to \Psi(s*x) = \top)$. Thus $\lambda x.\Psi(s * x)$ realizes ${\sim}B(|s|)$. Accordingly, by assumption on G it follows that $G_{|s|}(\lambda x.\psi(s * x))$ realizes U and thus also $B(n)$ (since $U \subseteq B(n)$). Thus $s * \lambda n.G_{|s|}(\lambda x.\psi(s * x))$ realizes $\forall n.B(n)$ and, accordingly, by assumption on Y it follows that $\Psi(s) = Y(s * \lambda n.G_{|s|}(\lambda x.\psi(s * x)))$ realizes U, i.e. $P(s)$ as desired.

Thus, since $\lambda G.\lambda Y.\mathsf{BR}(Y, G)(\langle\rangle)$ is proof-like it realizes the proposition

$$\forall B{:}\mathcal{P}_{\mathcal{K}}(N).\forall n.{\sim\sim}B(n) \to {\sim\sim}\forall n.B(n)$$

which, therefore, holds in $\mathcal{E}$ as claimed. $\square$

Notice that the form of bar induction used in the proof of Th. 9.1 is valid only because D^ω consists of **all** sequences in D and not just the computable ones.

Now we are ready to show that countable choice holds in $\mathcal{K}$.

Theorem 9.1 *For every object X in $\mathcal{K}$ the proposition*

$$\forall R{:}\mathcal{P}(N{\times}X).\, \forall n{:}N.\exists x{:}X.R(n, x) \to \exists f{:}X^N.\forall n{:}N.R(n, f(n))$$

hold in $\mathcal{K}$*.*

Proof: Since $\mathcal{K}$ is equivalent to the subtopos $\mathcal{E}_U$ of $\mathcal{E}$ consisting of sheaves for $j_U = {\sim} \circ {\sim}$ the problem reduces to showing that

$$\forall R{:}\mathcal{P}_{\mathcal{K}}(N{\times}X).\, \forall n{:}N.{\sim\sim}\exists x{:}X.R(n, x) \to {\sim\sim}\exists f{:}X^N.\forall n{:}N.R(n, f(n))$$

holds in $\mathcal{E}$. By Lemma 9.1 the implication

$$\forall n{:}N.{\sim\sim}\exists x{:}X.R(n,x) \to {\sim\sim}\forall n{:}N.\exists x{:}X.R(n,x)$$

holds in $\mathcal{E}$ and thus it suffices to show that

$$\forall R{:}\mathcal{P}_{\mathcal{K}}(N\times X).\, {\sim\sim}\forall n{:}N.\exists x{:}X.R(n,x) \to {\sim\sim}\exists f{:}X^N.\forall n{:}N.R(n,f(n))$$

holds in $\mathcal{E}$. This, however, holds since $\mathcal{E}$ validates countable choice and ${\sim\sim}$ commutes with implication. □

Thus, we have shown that $\mathcal{K}$ validates countable choice since X^N is isomorphic to $X^{N_{\mathcal{K}}}$ in $\mathcal{E}$.

Notice that for classical realizability models arising from countable term models one cannot apply the method we have used here because bar induction does not seem to be applicable since not every external sequence of terms can be represented by a term. Thus, for countable term models Krivine in [Kri03] introduced a `quote`-like construct for the purpose of realizing countable choice. Apparently, these two different methods are applicable under *mutually exclusive circumstances*. Whether countable choice holds in all realizability models is unknown up to now but one strongly suspects that the answer is negative!

9.2 Dependent Choice in $\mathcal{K}$

A topos with natural numbers object N validates the principle DC of *Dependent Choice* iff

$$\forall R{:}\mathcal{P}(N\times X\times X).\forall n{:}N.\forall x{:}X.\exists y{:}X.R(n,x,y) \to \\ \forall a{:}X.\exists f{:}X^N.\, f(0)=a \wedge \forall n{:}N.R(n,f(n),f(n{+}1))$$

holds for every object X of the topos. It is well know that $\mathcal{E}$ and actually every relative realizability topos validates DC. Unfortunately, the validity of Double Negation Shift in $\mathcal{E}$ is not sufficient for reducing validity of DC in $\mathcal{K}$ to its validity in $\mathcal{E}$. For this reason in Theorem 4 of [BO05] it is shown how to use modified bar recursion for realizing appropriate negative translations of DC. With some effort their proof can be adapted to $\mathcal{K}$. We leave the tedious details to the inclined reader. Notice, however, that Theorem 9.1 suffices already for showing that the infinite object $\Delta_{\mathcal{K}}(2)$ is also Dedekind infinite, i.e. that $\mathcal{K}$ validates the proposition $\exists f{:}\Delta_{\mathcal{K}}(2)^N(\forall n,m{:}N.f(n) \sim_{2_{\mathcal{K}}} f(m)$. However, this valid existential statement need not be witnessed by a global element of $\Delta_{\mathcal{K}}(2)^N$.

10 Is $\mathcal{K}$ 2-valued?

A proposition $A \in \Omega_{\mathcal{K}}$ is *valid in* $\mathcal{K}$ iff $A \cap P \neq \emptyset$. The topos $\mathcal{K}$ is 2-valued iff for every $A \in \Omega_{\mathcal{K}}$ either A or $\neg A$ has nonempty intersection with P.

Notice that for $t \in D$ we have $t \in P$ iff $\mathtt{tr}(t) \cap P^\omega = \emptyset$. Thus, if A holds in $\mathcal{K}$ then $A^{\perp\!\!\perp} \cap P^\omega = \emptyset$. If the reverse implication held as well then $\mathcal{K}$ would be

2-valued which can be seen as follows. Suppose A does not hold in $\mathcal{K}$. Then, due to our assumption, there exists $\vec{s} \in A^{\perp\!\!\perp} \cap P^\omega$ and thus $\lambda t.\vec{s} \in D^\omega.\, t(\vec{s})$ is an element of $P \cap \neg A$.

But if A is the biorthogonal closure of a countable subset of D we actually can reverse the implication.

Lemma 10.1 *If $A = \{t_n \mid n \in \omega\}^{\perp\!\!\perp\perp\!\!\perp}$ with $A^{\perp\!\!\perp} \cap P^\omega = \emptyset$ then $A \cap P \neq \emptyset$.*

Proof: W.l.o.g.[12] we assume that $t_{n+1}^{-1}(\top) \subseteq t_n^{-1}(\top)$ for all $n \in \omega$. We consider the countably branching tree $T = \bigcup_{n\in\omega} \{n\} \times \mathsf{tr}(t_n)$ where the ancestor of $\langle n{+}1, \vec{s}\rangle$ is the unique element $\langle n, \vec{r}\rangle$ with $\vec{r} \sqsubseteq \vec{s}$. Observe that for every $\vec{s} \in A^{\perp\!\!\perp}$ and $n \in \omega$ there is a unique $\vec{s}^{(n)} \in \mathsf{tr}(t_n)$ with $\vec{s}^{(n)} \sqsubseteq \vec{s}$. Thus, the minimal elements of $A^{\perp\!\!\perp}$ are precisely the suprema of the infinite paths in T, i.e. for every $\vec{s} \in \min(A^{\perp\!\!\perp})$ we have $\vec{s} = \bigsqcup_{n\in\omega} \vec{s}^{(n)}$. Thus, due to our assumption $A^{\perp\!\!\perp} \cap P^\omega = \emptyset$ every infinite path through T eventually leads out of P^ω. Let t be the element of D whose trace consists of those finite elements $\vec{s}$ of D^ω with $\vec{s}^{(n)} \notin P^\omega$ but $\vec{s}^{(k)} \in P^\omega$ for all $k < n$. Obviously, we have $t \in P$ and $\min(A^{\perp\!\!\perp}) \subseteq t^{-1}(\top)$. Thus $t \in A \cap P$ as desired. $\square$

In order to generalize this lemma to arbitrary propositions in $\mathcal{K}$ one could try to work with a well ordering of a biorthogonally closed subset A of D but then beyond stage ω the labels of the tree T are not finite anymore.

Another line of attack would be as follows. Suppose $A = A^{\perp\!\!\perp\perp\!\!\perp}$ such that $\mathsf{tr}(t) \cap P^\omega \neq \emptyset$ for all $t \in A$. Notice that the (upward closures) of the sets $\mathsf{tr}(t) \cap P^\omega$ with $t \in A$ form a filter w.r.t. the Smyth ordering. But, alas, we do not know how to prove that the intersection of the elements of this filter has to be non-empty.

On the other hand we do not know any particular biorthogonally closed subset of D which does not already arise as the biorthogonal of a countable subset. In particular, we may replace any proposition A with the biorthogonal closure of the intersection of A with the computable elements of A. Maybe this does not make any difference for propositions A arising from the interpretation of a closed formula in the language of set theory.

11 Summary

We have shown that a new boolean non-Grothendieck topos $\mathcal{K}$ arises from a canonical model of λ-calculus with control in the category **Coh** of coherence spaces and stable functions. We have shown that $\mathcal{K}$ validates all true sentences of first order arithmetic and also countable (and dependent) choice.

We have also observed that the model constructions collapses to the ground model **Set** when starting form the canonical model of λ-calulus with control in Scott domains where as usual the culprit is parallel-or.

[12] This can be achieved easily since $\sqcap : \Sigma \times \Sigma \to \Sigma$ is stable.

There are still quite a few open questions about the topos $\mathcal{K}$ arising from the stable model of λ-calulus with control. One would like to see a conrete example of a set-theoretic statement holding in **Set** but not in $\mathcal{K}$. We suspect that AC, the full axiom of choice, is such an example but have not been able yet to verify this. Moreover, one would like to know whether every closed formula in the language of set theory is decided by $\mathcal{K}$.

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