Identity Types vs. Weak ω-Groupoids

Some Ideas and Problems

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Identity Types (1)

are the most intriguing concept of intensional Martin-Löf type theory (ITT). They are given by the rules

$$\frac{\Gamma \vdash A}{\Gamma, x, y{:}A \vdash \mathsf{Id}_A(x,y)} \text{(Id-}F\text{)}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash r_A(x) : \mathsf{Id}_A(x,x)} \text{(Id-}I\text{)}$$

$$\frac{\Gamma, x, y{:}A, z : \mathsf{Id}_A(x,y) \vdash C(x,y,z) \qquad \Gamma, x{:}A \vdash d : C(x,x,r_A(x))}{\Gamma, x, y{:}A, z : \mathsf{Id}_A(x,y) \vdash J((x)d)(z) : C(x,y,z)} \text{(Id-}E\text{)}$$

together with the conversion rule

$$J((x)d)(r_A(t)) = d[t/x]$$

and motivated by the intention that *all* concepts appear as inductively defined (families of) types.

(c.f. the axiomatization of equality in predicate logic by the axiom $\top \vdash_{\{x\}} x = x$ and the rule $\top \vdash_{\{x\}} \varphi(x,x)$ iff $x = y \vdash_{\{x,y\}} \varphi(x,y)$)

Identity Types (2)

Using J one can define operations

$cmp_A \in (\Pi x, y, z{:}A)\,\mathrm{Id}_A(x,y)\to\mathrm{Id}_A(y,z)\to\mathrm{Id}_A(x,z)$

$inv_A \in (\Pi x, y{:}A)\,\mathrm{Id}_A(x,y) \to \mathrm{Id}_A(y,x)$

validating (where we write id_x for $r_A(x)$)

(a) $(\Pi x, y, z, u{:}A)$
$(\Pi f{:}\mathrm{Id}_A(x,y))(\Pi g{:}\mathrm{Id}_A(y,z))(\Pi f{:}\mathrm{Id}_A(z,u))$
$\mathrm{Id}_{\mathrm{Id}_A(x,u)}(cmp(f, cmp(g,h)), cmp(cmp(f,g),h))$

(b) $(\Pi x, y{:}A)\ \mathrm{Id}(cmp(id_x, f), f) \wedge \mathrm{Id}(cmp(g, id_y), g)$

(c) $(\Pi x, y{:}A)(\Phi f{:}\mathrm{Id}_A(x,y))$
$\mathrm{Id}(cmp(f, inv(f)), id_x) \wedge \mathrm{Id}(cmp(inv(f), f), id_y)$

rendering type A as an **internal groupoid where the groupoid equations hold only in the sense of propositional equality**, i.e. for instance (a) means that there is a term $assoc_A(f,g,h)$ of type
$\mathrm{Id}_{\mathrm{Id}_A(x,u)}(cmp(f, cmp(g,h)), cmp(cmp(f,g),h))$
which may be thought of as a **2-cell** in the sense of *higher dimensional categories*.

Identity Types (3)

In early 1990ies I observed that one can prove that

$$(\Pi A{:}\mathsf{Set})(\Pi x, y{:}A)(\Pi{:}f, g{:}\mathsf{Id}_{\mathsf{Id}_A(x,y)}(f, g))$$

i.e. *proof irrelevance for equality proofs* (PIE) using the following natural extension of MLTT

$$\frac{\Gamma, x{:}A, z : \mathsf{Id}_A(x, x) \vdash C(x, z) \qquad \Gamma, x{:}A \vdash d : C(x, r_A(x))}{\Gamma, x{:}A, z : \mathsf{Id}_A(x, x) \vdash K((x)d)(z) : C(x, z)} \ (\mathsf{Id}\text{-}E')$$

together with the conversion rule

$$K((x)d)(r_A(t)) = d[t/x]$$

Using K one easily shows that $(\Pi x{:}A)(\Pi f{:}\mathsf{Id}_A(x, x))\mathsf{Id}_{\mathsf{Id}_A(x,x)}(id_x, f)$ and thus $(\Pi x{:}A)(\Pi f, g{:}\mathsf{Id}_A(x, x))\mathsf{Id}_{\mathsf{Id}_A(x,x)}(f, g)$ from which it follows using J that $(\Pi x, y{:}A)(\Pi f, g{:}\mathsf{Id}_A(x, y))\mathsf{Id}_{\mathsf{Id}_A(x,y)}(f, g)$.

NB In implementations of MLTT like in ALF or AGDA etc. their *pattern matching* facilities allows on to derive K very easily. The extension preserves good properties like normalization, decidability of type checking etc.

Groupoid Model of ITT

In 1994 [HS95] M. Hofmann and I constructed a groupoid model for ITT where K does not exists and (a)-(c) hold in the sense of propositional equality. The **key idea** was to interpret **types as groupoids** and **families of types as fibrations of groupoids** and

$$\mathrm{Id}_A(x, y) \quad \text{as} \quad A(x, y)$$

which may contain more than one element if the groupoid is not posetal. Thus

PIE fails in the groupoid model!

Already in [HS95] it was observed that this (deviating?) interpretation has the benefit that

> the **bureaucracy of identity types** forces one to **check all coherence conditions** when reasoning up to isomorphism

i.e. when treating 'isomorphic' as 'equal' (as categorists like to do) which sometimes is a source of mistakes when done naively!

Towards ω-Groupoids

Already in [HS95] it was observed that ω-groupoids might be more appropriate since in ITT the types $\mathrm{Id}_A(x, y)$ are groupoids themselves. We also observed that strict ω-groupoids are not sufficient either because in ITT the conditions (a), (b) and (c) do **not hold in the sense of judgemental equality** but **only in the sense of propositional equality**, i.e. that **weak ω-groupoids are more appropriate**.

A bit later Category Theory saw the advent of **Higher Dimensional Categories** (hdc's) and **Higher Dimensional Groupoids** (hdg's). We suspected that they might be relevant for modelling ITT faithfully (but there was a "cultural gap").

Towards ω-Groupoids (2)

However, **models of ITT are strict categories** (i.e. not “weak” themselves). It rather is the case that **types are internal higher dimensional groupoids** and maps are **internal functors of hdg’s**.
Moreover these internal hdg’s are more specific in the following respects.

(1) The internal groupoids associated with types in ITT are a bit “less weak” than ω-groupoids since there is a **choice** of **composition** and **inversion** and they are not just given by total relations. Moreover, these choices are given internally by the maps cmp and inv.
Can be possibly overcome for Kan complexes in the category $\mathcal{SS}$ of simplicial sets.

(2) All this structure is **stable under substitution**, i.e. stable under pullbacks aka **Beck-Chevalley condition** (BC).

Categ. Models of TT (1)

(1) Contexts and substitutions are modeled by a category $\mathcal{C}$.

(2) Families of types are modeled by a discrete fibration $P : \mathcal{T} \to \mathcal{C}$ together with a cartesian functor $C : P \to P_{\mathcal{C}}$ (where $P_{\mathcal{C}} = \partial_1 : \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ is the fundamental fibration for $\mathcal{C}$) called *comprehension*.
For $f : \Delta \to \Gamma$ in $\mathcal{C}$ and $A \in P(\Gamma)$ we thus obtain (by applying C) a *canonical pullback*

$$\begin{array}{ccc} \Delta.f^*A & \xrightarrow{q(f,A)} & \Gamma.A \\ {\scriptstyle p(f^*A)}\downarrow & \lrcorner & \downarrow{\scriptstyle p(A)} \\ \Delta & \xrightarrow[f]{} & \Gamma \end{array}$$

writing $p(A) : \Gamma.A \to \Gamma$ for $C(A)$.

NB Since $C : P \to P_{\mathcal{C}}$ is a (cartesian) functor the **choice of canonical pullbacks** is **functorial**.

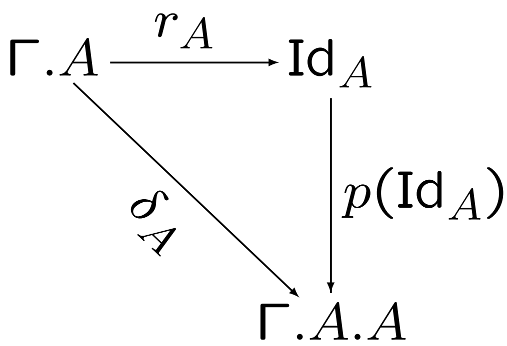
Categ. Models of TT (2)

If $A \in P(\Gamma)$ we write A^*A for $p(A)^*A$ and $\Gamma.A.A$ as abbreviation for $\Gamma.A.A^*A$. Further we write δ_A for the canonical map from $\Gamma.A$ to $\Gamma.A.A$.

An **identity type** for A is given by

(1) $\mathsf{Id}_A \in P(\Gamma.A.A)$

(2) a map r_A with



(3) for $C \in P(\mathsf{Id}_A)$ and $d : \Gamma \to \Gamma.A^*A.\mathrm{Id}_A.C$ with $p(C) \circ d = r_A$ a section $J(d)$ of $p(C)$ with $J(d) \circ r_A = f$.

These choices are stable under substitution (BC), i.e. for $f : \Delta \to \Gamma$ it holds that

$$f^*\mathsf{Id}_A = \mathsf{Id}_{f^*A} \quad f^*r_A = r_{f^*A} \quad f^*J(d) = J(f^*d)$$

Modified Assemblies (1)

were defined in my Habil. Thesis (1993) [Str93] as the $\neg\neg$-separated objects of mr-$\mathbf{Set} = \mathbf{Set}{\downarrow}\Gamma$ where $\Gamma = \mathcal{E}\!f\!f(1,-) : \mathcal{E}\!f\!f \to \mathbf{Set}$.
More concretely, this category can be described as so-called **modified assemblies**. Objects are triples $X = (|X|, \Vdash_X, P_X)$ where $(|X|, \Vdash_X)$ is an assembly and $P_X \subseteq |X|$.
Idea : $|X|$ are the **potential** objects and P_X are the **actual** objects. Morphisms from X to Y are assembly morphisms $f : (X, \Vdash_X) \to (Y, \Vdash_Y)$ *preserving actual objects*, i.e.

$$\begin{array}{ccc} P_X & \cdots\cdots\to & P_Y \\ \cap\downarrow & & \downarrow\cap \\ |X| & \xrightarrow{\;f\;} & |Y| \end{array}$$

Let $\Delta(2)$ be the assembly with underlying set $\{0, 1\}$ and $j \Vdash_{\Delta(2)} i$ iff $i = j$.
We define identity types as follows

$$\mathrm{Id}_X(x, y) = (\Delta(2), \{1\}) \qquad \text{if } x = y$$

$$\mathrm{Id}_X(x, y) = (\Delta(2), \emptyset) \qquad \text{if } x = y$$

Modified Assemblies (2)

For $f, g : X \to Y$ we have
$x{:}X \vdash f(x) = f(y) : Y$ holds iff $f = g$ and
$x{:}X \vdash \mathrm{Id}_Y(f(x), f(y))$ holds iff $f_{|P_X} = gf_{|P_X}$
Thus propositional equality does **not entail** judgemental equality.

This makes it possible to meet the following criteria of being a "truely non-extensional" model for ITT + K (see [Str93] for a verification)

(C1) $x, y{:}A, z{:}\mathrm{Id}_X(x, y) \vdash x = y : A$ fails

(C2) $x, y{:}A, z{:}\mathrm{Id}_X(x, y) \vdash B(x) = B(y)$ fails

(C3) $\vdash p : \mathrm{Id}_A(t, s)$ implies $\vdash t = s : A$

for arbitrary $A : \mathrm{Set}$ and $B : A \to \mathrm{Set}$.

However, one may show that K can be interpreted since it holds in mr-$\mathbf{Set}$ that $p, q : \mathrm{Id}_X(x, y) \vdash \mathrm{Id}_{\mathrm{Id}_X(x,y)}(p, q)$.

NB Obviously (C3) is not generally valid in groupoid or homotopy models!

Weak ω-Groupoids (1)

Definitions of hdc's and hdg's are fairly complex. Personally, I find A. Joyal's approach to weak ω-groupoids and "quasi-categories" (ω-categories where n-cells for $n \geq 2$) the most understandable one because it takes place within the category $\mathcal{SS} = \widehat{\mathbf{\Delta}}$ of *simplicial sets*.

Let $\mathbf{\Delta}$ be the category of finite nonempty ordinals and order preserving maps between them.

We write $\Delta[n]$ or simply $[n]$ for $\{0, 1, \ldots, n\}$. The maps of $\mathbf{\Delta}$ are generated by the morphisms

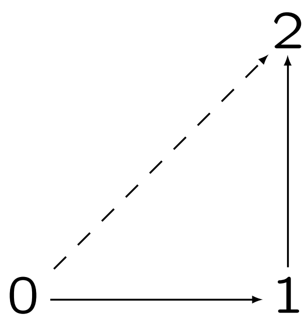
$$d_n^i : [n{-}1] \to [n] \qquad s_n^i : [n] \to [n{-}1]$$

where the first one is monic and omits i and the second one is epic and "repeats" i.

For $0 \leq i \leq n$ let $\partial_i[n]$ be the subobject of $[n]$ consisting all maps $u : [m] \to [n]$ with $i \not\in \mathrm{im}[u]$. We write $\partial[n]$ for the subobject $\bigcup_{i=0}^n \partial_i[n]$ of $[n]$. $\partial[n]$ is called the *boundary* of $[n]$ (or $(n{-}1)$-*sphere*).

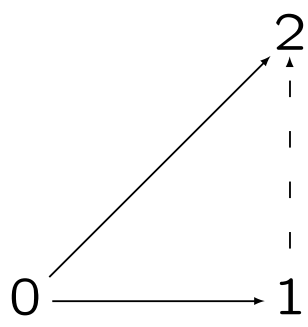
Weak ω-Groupoids (2)

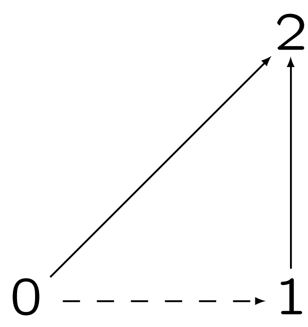
A k-**horn** in $[n]$ for $0 \leq k \leq n$ is the subobject $\Lambda^k[n] = \bigcup_{i \neq k} \partial_i[n]$ of $[n]$. Thus $\Lambda^k[n]$ is the union of all $(n-1)$-faces of $[n]$ containing the node k. E.g. $\Lambda^1[2]$ can be depicted as



where the omitted faces are indicated by broken lines.

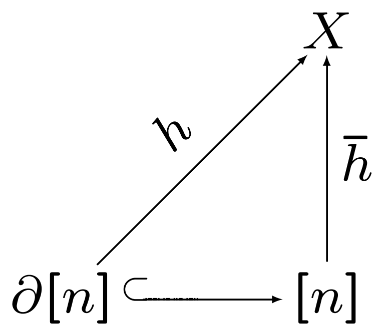
$\Lambda^k[n]$ is an **inner horn** iff $0 < k < n$. $\Lambda^1[2]$ is an inner horn as opposed to the horns $\Lambda^0[2]$ and $\Lambda^2[2]$ depicted as





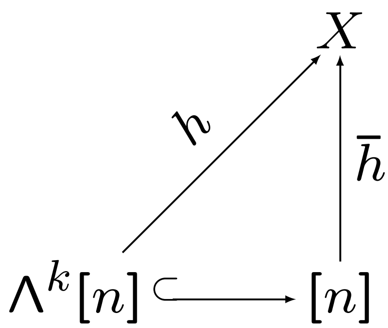
Weak ω-Groupoids (3)

Let X be a simplicial set. An n-**diagram** in X is a morphism $h : \partial[n] \to X$ which **commutes** iff there is a $\bar{h} : [n] \to X$ making



commute.

A **quasi-category** is a simplicial set such that every **inner** horn $h : \Lambda^k[n] \to X$ in X can be extended to a $\bar{h} : [n] \to X$ making

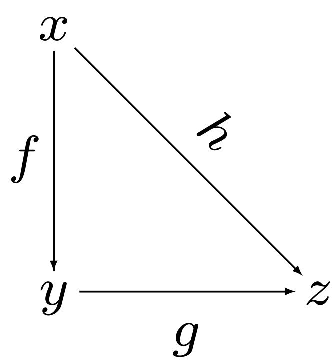


commute.

A simplicial set X is a **quasi-groupoid** (aka **Kan complex**) iff the above extension property holds for **all** horns (and not just the inner ones).

Weak ω-Groupoids (4)

Let X be a quasi-category. 1-cells in X are $f \in X(1)$ whose source and target are given by $x = X(d_1^1)(f)$ and $y = X(d_1^0)(f)$, resp., for which we write $f : x \to y$.
For $x \in X(0)$ we write id_x for $X(s_1^0)(x)$.
If $f : x \to y$ and $g : y \to z$ then extension along $\Lambda^1[2] \hookrightarrow [2]$ gives rise to a 2-cell α



giving rise to a (non-unique) composition. If there exists such a 2-cell we write $h \sim g \cdot f$. Factoring 0- and 1-cells of X modulo the congruence generated by $\sim$ we get the **fundamental category** cat(X) **associated with** X.

Weak ω-Groupoids (5)

If X is a quasi-groupoid by extendability along the inclusions of $\Lambda^0[2]$ and $\Lambda^2[2]$ into $[2]$ we get for every $f : x \to y$ a $g : y \to x$ with $id_x \sim g \cdot f$ and a $g' : y \to x$ with $id_y \sim f \cdot g'$.
A. Joyal has shown that a quasi-category X is a Kan complex iff cat(X) is a groupoid in which case cat(X) is called the **fundamental groupoid** of X also denoted by $\pi_1(X)$.

Idea Thus it seems natural to interpret intensional type theory ITT in $\mathcal{SS}$ where types are given by quasi-groupoids.
A notion of *families of types* is given by the so-called *Kan fibrations* as arising in the classical homotopy (or Quillen) structure on simplicial sets which we recall next.

Warning Even if this goes through there is the problem that type constructions of ordinary ITT will not lead out of **discrete** simplicial sets (constant presheaves over Δ). Thus, equality on "ordinary" types of ITT will still be extensional! This defect was already observed for the ordinary groupoid model of Hofmann and Streicher.

Quillen structure on $\mathcal{SS}$ (1)

Quasi-groupoids or Kan complexes play an important role in *simplicial homotopy* theory. A **Kan fibration** is a map $p : X \to Y$ in $\widehat{\mathbf{\Delta}}$ such that for every commuting square

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ [n] & \xrightarrow[g]{} & Y \end{array}$$

there exists **some** (not necessarily unique) $h : [n] \to X$ making the diagram

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & \nearrow{\scriptstyle h} & \big\downarrow{\scriptstyle p} \\ [n] & \xrightarrow[g]{} & Y \end{array}$$

commute. (**Notation** $i \perp p$)
Obviously X is a Kan complex iff the terminal projection $X \to 1$ is a Kan fibration.

Quillen structure on $\mathcal{SS}$ (2)

The **canonical Quillen structure** on $\mathcal{SS}$ is given by $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ where

$\mathcal{C}$ = class of monomorphisms

$\mathcal{W}$ = class of weak equivalences

$\mathcal{F}$ = class of Kan fibrations

where $w : X \to Y$ is called a weak equivalence iff $\mathcal{SS}(w, Z)$ is onto for all Kan complexes. We don't give here the axioms for Quillen structures but recall just the following properties.

(1a) $i \perp p$ for all $i \in \mathcal{C} \cap \mathcal{W}$ and $p \in \mathcal{F}$

(1b) $i \perp p$ for all $i \in \mathcal{C}$ and $p \in \mathcal{F} \cap \mathcal{W}$

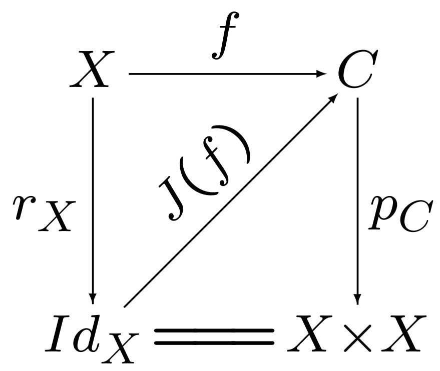
(2a) every morphism f in $\mathcal{SS}$ factors as $f = p \cdot i$ for some $i \in \mathcal{C} \cap \mathcal{W}$ and $p \in \mathcal{F}$

(2a) every morphism f in $\mathcal{SS}$ factors as $f = p \cdot i$ for some $i \in \mathcal{C}$ and $p \in \mathcal{F} \cap \mathcal{W}$

Quillen structure on $\mathcal{SS}$ (3)

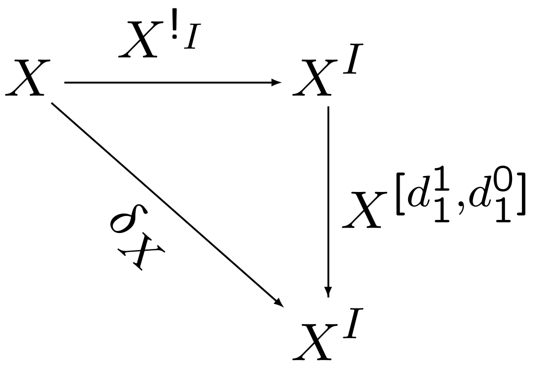
By (2a) for every Kan complex X the map $\delta_X : X \to X \times X$ factors as $X \stackrel{r_X}{\to} \mathrm{Id}_X \stackrel{p}{\to} X \times X$ with r_X a monic weak equivalence and p_X a Kan fibration.

By (1a) it follows that for every Kan fibration $p_C : C \to X \times X$ and $f : X \to C$ with $p_C \cdot f = r_X$ there exists a map $J(f) : X \to C$ making



commute since $r_X \perp p_C$.

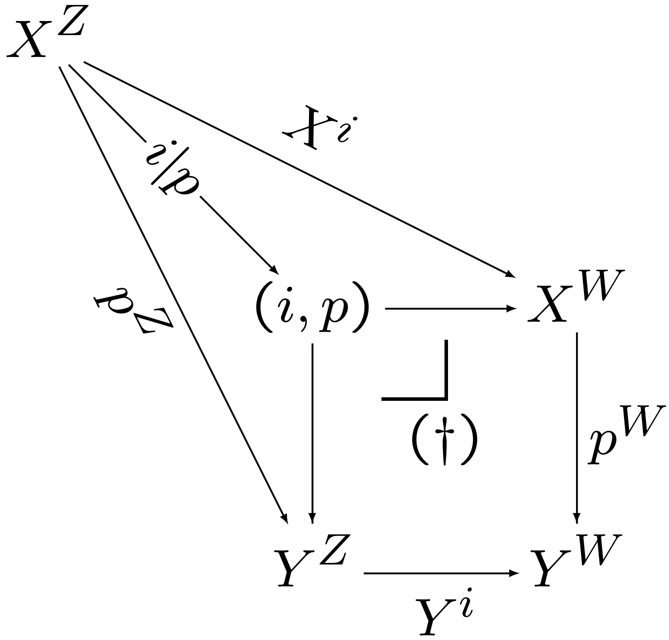
M. Warren observed that a factorization of δ_X as required by (2) can be obtained as



where $I = \Delta[1]$

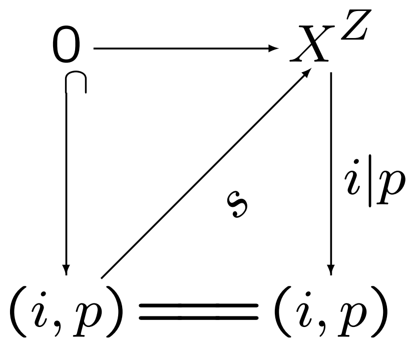
Internalising J (1)

Given a Kan fibration $p : X \to Y$ and a map $i : W \to Z$ in $\mathcal{C} \cap \mathcal{W}$ one may consider the mediating arrow $i|p$ in



and observe that $i \perp p$ is equivalent to $\mathcal{SS}(1, i|p)$ being surjective.

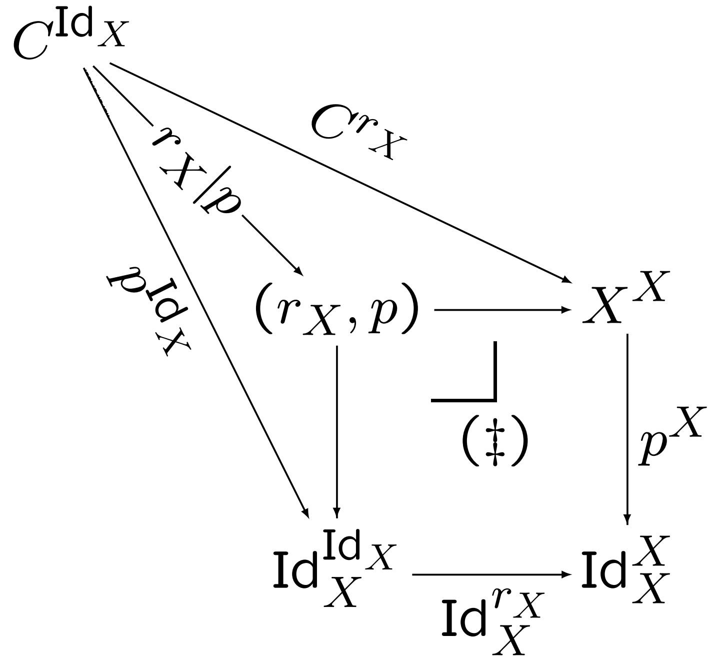
However, it can be shown (JT book) that $i|p$ is in $\mathcal{W} \cap \mathcal{F}$ and accordingly a split epi since



because all $0 \hookrightarrow i|p$ is monic and thus in $\mathcal{C}$.

Internalizing J (2)

Now instantiating (†) with $i = r_X$ and Kan fibration $p : C \to \mathsf{Id}_X$ we get



and a section s of $r_X|p$.

Now we get

$$J = \ulcorner id \urcorner^* s$$

where $\ulcorner id \urcorner = \Lambda(1 \times \mathsf{Id}_X \xrightarrow{\pi'} \mathsf{Id}_X)$.

Notice that $\ulcorner id \urcorner^* C^{\mathsf{Id}_X}$ is Sect(p), the object of sections of p.

Conclusion

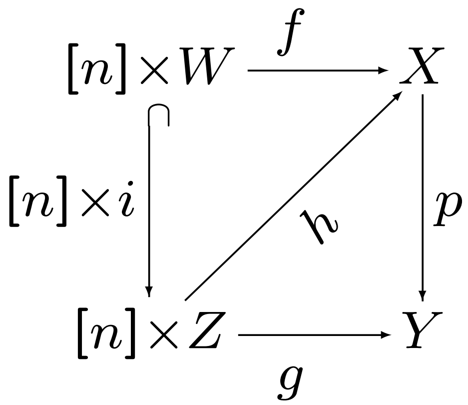
- Kan fibrations in simplicial sets are expected (Warren's Thesis) to provide a model for intensional identity types.

- However, the interpretation of syntactically definable types of ITT stay within **discrete** simplicial sets where everything is extensional, i.e. r_X is isomorphic to δ_X for discrete X.

- Maybe simplicial sets within mr-$\mathbf{Set}$ lead out of this dilemma ?
(**But** I don't see how to avoid PIE holding for syntactically definable types!)

Serre and Hurewicz fibr's

An intermediary condition would be that $i|p$ is epic. More explicitly this means that for every representable object $[n]$ and maps $f : [n]\times W \to X$ and $g : [n]\times Z \to Y$ with $p\cdot f = g\cdot i$ there exists a map $h : [n] \times Z \to X$ making the diagram



commute. This reminds one of the definition of a **Serre fibration** where the representable objects play $[n]$ the role of n-simplices.

If $i|p$ is split epi then the above property holds not only for representable objects $[n]$ but for arbitray simplicial sets U which condition reminds one of a **Hurewicz fibration** but, actually, is somewhat stronger since for Hurewicz fibrations the choice of h need not be functorial in U.