

Observations on Fibered Toposes

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1 Toposes Fibered over a Base Topos

Let \mathcal{S} be a (base) topos. A fibration $P : \mathcal{X} \rightarrow \mathcal{S}$ a fibration is a fibration of toposes iff all fibers are toposes and all reindexing functors are logical.

Theorem 1.1 *A fibration $P : \mathcal{X} \rightarrow \mathcal{S}$ of toposes has internal sums iff \mathcal{X} is a topos and P is a logical functor.*

Proof: By a theorem of Jibladze from 1988 a fibered topos $P : \mathcal{X} \rightarrow \mathcal{S}$ has internal sums iff P is equivalent to the fibration $P_F = \partial_1 : \mathcal{E} \downarrow F \rightarrow \mathcal{S}$ for some finite limit preserving functor $F : \mathcal{S} \rightarrow \mathcal{E}$ between toposes. Notice that F is unique up to isomorphism since $FI = \coprod_I 1_I$. From the theory of Artin glueing one knows that $\mathcal{E} \downarrow F$ is a topos and P_F is logical.

Suppose that $P : \mathcal{X} \rightarrow \mathcal{S}$ is a fibration of toposes with \mathcal{X} a topos and P logical (or just preserving the cartesian closed structure). Since P is in particular a fibration of finite limit categories it has a full and faithful right adjoint 1 picking terminal objects in each fiber. Thus $P \vdash 1$ is an injective geometric morphism. Proposition A.4.5.1 of Johnstone's *Elephant* tells us that P is logical iff $P \dashv 1$ is equivalent to $U^* \dashv \Pi_U : \mathcal{X}/U \rightarrow \mathcal{X}$ for some subterminal U . Notice that U is isomorphic to $\Sigma_U 1_{\mathcal{X}}$ where $\Sigma_U \dashv U^*$. From the theory of Artin glueing one knows that the open subtopos $U^* \vdash \Pi_U$ has a closed complement $j : \mathcal{E} \hookrightarrow \mathcal{X}$ induced by the closure operator $U \vee (-)$ on \mathcal{X} and that the fibration $U^* : \mathcal{X} \rightarrow \mathcal{X}/U$ is equivalent to the fibration $P_F = \partial_1 : \mathcal{E}/F \rightarrow \mathcal{S}$ where $F = j^* \Pi_U$. But since F preserves finite limits the fibration P_F equivalent to P has internal sums. \square

Since in the proof of the backwards direction we have just used that P preserves finite limits and exponentials we get the

Corollary 1.1 *A fibration $P : \mathcal{X} \rightarrow \mathcal{S}$ of toposes has internal sums iff \mathcal{X} is a topos and P preserves finite limits and exponentials.*

2 A topos \mathcal{E} has small sums iff $\mathbf{Fam}(\mathcal{E})$ is a topos

For every topos \mathcal{E} the fibration $Fam(\mathcal{E}) : \mathbf{Fam}(\mathcal{E}) \rightarrow \mathbf{Set}$ is a fibration of toposes.

Theorem 2.1 *For an elementary topos \mathcal{E} the category $\text{Fam}(\mathcal{E})$ is cartesian closed iff \mathcal{E} has small sums.*

Proof: If \mathcal{E} has small sums then $\text{Fam}(\mathcal{E})$ is equivalent to $\mathcal{E}\downarrow\Delta$ where $\Delta(I) = \coprod_I 1_{\mathcal{E}}$. Since $\Delta : \mathbf{Set} \rightarrow \mathcal{E}$ preserves finite limits $\mathcal{E}\downarrow\Delta$ is a topos and thus in particular cartesian closed.

Suppose $\text{Fam}(\mathcal{E})$ is cartesian closed. Since it has a subobject classifier anyway it is also a topos. As observed by Peter Johnstone the fibration $\text{Fam}(\mathcal{E})$ is equivalent to $U^* : \text{Fam}(\mathcal{E}) \rightarrow \text{Fam}(\mathcal{E})/U$ for $U = (1, (0))$ (the initial object in the fiber over 1) since in \mathcal{E} every morphism to 0 is an iso. Thus $\text{Fam}(\mathcal{E})$ is logical since U^* is logical from which it follows by Theorem 1.1 that $\text{Fam}(\mathcal{E})$ has internal sums, i.e. \mathcal{E} has small sums. \square

Thus $\text{Fam}(\mathcal{E})$ is not cartesian closed if \mathcal{E} is the free topos (with nno) or a nontrivial realizability topos though $\text{Fam}(\mathcal{E})$ certainly is a fibered topos.