Zawadowski's Cartesian Bifibrations

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If \( \mathcal{C} \) is a category with finite limits then \( P_\mathcal{C} = \partial_1 : \mathcal{C} \downarrow \mathcal{C} \to \mathcal{C} \) is a bifibration such that

1. \( P_\mathcal{C} \) is a fibration of finite limit categories
2. for all \( u : J \to I \) the functor \( u_! \) (left adjoint to the reindexing functor \( u^* \)) preserves pullbacks
3. for all \( u : J \to I \) the unit and counit of the adjunction \( u_! \dashv u^* \) are cartesian\(^1\).

If \( F : \mathcal{B} \to \mathcal{C} \) is a functor between categories with finite limits and \( F \) preserves terminal objects then \( F^* P_\mathcal{C} \) is a bifibration over \( \mathcal{B} \) still satisfying the conditions (1)-(3) above. We call such fibrations \textit{cartesian bifibrations}.\(^2\)

We now show that every cartesian bifibration \( P : \mathcal{X} \to \mathcal{B} \) is equivalent to one of the form \( F^* P_\mathcal{C} \) for some terminal object preserving functor \( F : \mathcal{B} \to \mathcal{C} \) between finite limit categories. For this purpose the following Lemma will be crucial.

**Lemma** If \( P : \mathcal{X} \to \mathcal{B} \) is a cartesian bifibration and

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi} & Z \\
\downarrow{\alpha} & & \downarrow{\beta} \\
X & \xrightarrow{\text{cocart.}} & u_! X \\
\end{array}
\]

\[\tau\]

1. a natural transformation \( \tau : F \Rightarrow G \) is cartesian iff for all \( \alpha : X \to Y \) the square

\[
\begin{array}{ccc}
FX & \xrightarrow{\tau_X} & GX \\
\downarrow{F\alpha} & & \downarrow{G\alpha} \\
FY & \xrightarrow{\tau_Y} & GY \\
\end{array}
\]

is cartesian, i.e. a pullback

2. They were introduced by M. Zawadowski in his paper \textit{Lax Monoidal Fibrations} (2009).
is a commuting square in $X$ with $\alpha$ and $\beta$ vertical and $\varphi$ cocartesian then this square is a pullback if and only if $\psi$ is cocartesian.

**Proof.** Suppose $\psi$ is cocartesian. Consider the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\eta_Y} & u^*u_1 Y \\
\downarrow \alpha & & \downarrow \beta \\
X & \xrightarrow{\eta_X} & u^*u_1 X
\end{array}
\]

where the bottom and top arrow are $\varphi$ and $\psi$, respectively. Since both squares are pullbacks so is the rectangle.

Now suppose that (*) is a pullback. Consider the diagram

\[
\begin{array}{ccc}
u_1 Y & \xrightarrow{u_1 \gamma} & wu^* Z \\
\downarrow \text{cocart.} & & \downarrow \text{cocart.} \\
Y & \xrightarrow{\gamma} & u^* Z \\
\downarrow \alpha & & \downarrow \beta \\
X & \xrightarrow{\eta_X} & u^*u_1 X \\
\end{array}
\]

where the bottom and top arrow of the rectangle are $\varphi$ and $\psi$, respectively. Thus, for showing that $\psi$ is cocartesian it suffices to show that $\varepsilon_Z \circ u_1 \gamma$ is an isomorphism. For showing the latter consider the diagram

\[
\begin{array}{ccc}
u_1 Y & \xrightarrow{u_1 \gamma} & wu^* Z \\
\downarrow u_1 \alpha & & \downarrow \beta \\
u_1 X & \xrightarrow{u_1 \eta_X} & wu^*u_1 X \\
\end{array}
\]

where the left square is a pullback because $u_1$ preserves pullbacks and the right square is a pullback since $\varepsilon$ is cartesian. Thus, the rectangle is a pullback, too. Since the bottom arrow of the rectangle is an identity arrow it follows that the top arrow of the rectangle is an isomorphism as claimed.

From this Lemma it follows that $P$ is equivalent to $\Delta^* P_{X_1}$ where $\Delta : B \to X_1$.

\[2\]
is given by

\[
\begin{array}{ccc}
X & \xrightarrow{\psi_X} & \Delta(X) \\
\downarrow{\alpha} & & \downarrow{\Delta(\alpha)} \\
Y & \xrightarrow{\psi_Y} & \Delta(Y)
\end{array}
\]

where $\psi_X$ and $\psi_Y$ are cocartesian over terminal projections.

Notice that $\Delta$ preserves pullbacks iff cocartesian arrows are stable under pullbacks along cartesian arrows, i.e. $P$ has internal sums.